Final exam
APPM 5440 Fall 2014 Applied Analysis

Date: Tuesday, Dec. 15 2014, 10:30 AM to 1 PM
You may assume all vector spaces are over the real field unless otherwise specified.
Your proofs may use any major result discussed in class (if you are unsure, please ask).
Total points possible: 107. **N.B.** Unlike the homeworks, the grades may be curved.
Note that points are *not* distributed according to difficulty, e.g., you may wish to save problems 9 and 10 for last since they are not worth many points.

Problem 1: Compactness (12 points, 2 points each)

(a) State the *topological* definition of a **compact** set

(b) State the definition of a **sequentially compact** set

(c) State the simplified definition of a compact set in a finite-dimensional metric vector space

(d) State a definition of a **pre-compact** set

(e) State a definition of a **compact** linear operator on a normed vector space

(f) Give an example of a set that is bounded but not totally bounded

Problem 2: Major theorems (14 points, 2 points each) State the following theorems. You may choose *one* of these to skip (please indicate which one you do not want graded)

(a) State the Arzelà-Ascoli theorem

(b) State the Weierstrass Approximation theorem

(c) State the Banach contraction mapping theorem

(d) State the Hahn-Banach theorem (any version)

(e) State the open-mapping theorem

(f) State Grönwall's inequality

(g) State Bessel's inequality regarding an orthonormal set $U = (u_n)_{n \in \mathbb{N}}$ (you do not need to consider uncountable sets)

(h) State at least three equivalent definitions of an orthonormal basis (again, you do not need to consider uncountable sets)

Problem 3: (16 points, 2 points each) Weak topologies and convergence. Let X be a normed linear space. You do not need to justify your answer.

(a) What does it mean to say a set $F \subset X$ is weakly closed?

(b) If F is weakly closed, is it necessarily also closed?

(c) If F is closed, is it necessarily also weakly closed?

(d) If $(x_n) \subset X$ and $x_n \rightharpoonup x$, then does $\|x_n\| \to \|x\|$?

(e) Are weak convergence and strong convergence the same if $X = \mathbb{R}$ (with the standard Euclidean norm)?

(f) Let X be a Banach space and X^* its topological dual. What is weak-* convergence?

(g) When is weak-* convergence equivalent to weak convergence?

(h) Let X and Y be Banach spaces. What does it mean to say that a sequence (T_n) in $\mathcal{B}(X, Y)$ *converges strongly* (*not* uniformly)? i.e., strong operator convergence.

Problem 4: (15 points) Let $(f_n) \subset C(I)$, and consider

$$\lim_{n\to\infty} \int_I f_n(x)\, dx \stackrel{?}{=} \int_I f(x)\, dx.$$

When is this necessarily true? If it is true, prove it, and if it is false, provide a counter-example.

(a) (5 points) f_n converges uniformly to f, and $I = [a, b]$.

(b) (5 points) f_n converges point-wise to f, and $I = [a, b]$.

(c) (5 points) f_n converges uniformly to f, and $I = \mathbb{R}$.

Problem 5: (10 points) Prove Lemma 1.63: a sequentially compact metric space X is separable. You may use other relevant theorems in your proof.

Problem 6: (5 points) Let X be a space of points, and $\mathcal{T}_d$ be the discrete topology $\mathcal{P}(X)$, and $\mathcal{T}_t$ be the trivial topology $\{\emptyset, X\}$. Consider the identity map $\mathcal{I} : (X, \mathcal{T}_t) \to (X, \mathcal{T}_d)$. Is this map continuous? Briefly prove your answer.

Problem 7: (10 points) Let X be a normed linear space. A series $\sum x_n$ in X is *absolutely convergent* if $\sum \|x_n\|$ converges to a finite value in $\mathbb{R}$.

(a) (7 points) Prove that if X is a Banach space, then every absolutely convergent series converges

(b) (3 points) Conversely, prove that if every absolutely convergent series converges, then X must be Banach.

Problem 8: (15 points) Show that for $1 \le p < \infty$, then $(\ell^p)^* = \ell^q$ (in the sense that the two spaces are isometrically isomorphic) for $1/q = 1 - 1/p$, and that $(\ell^\infty)^* \neq \ell^1$. You may take the field to be $\mathbb{R}$, and you may use standard theorems and inequalities that have been discussed in class. Specifically,

(a) (5 points) Show $(\ell^p)^* \subset \ell^q$ for $1 \le p < \infty$.

(b) (5 points) Show $\ell^q \subset (\ell^p)^*$ for $1 \le p < \infty$.

(c) (3 points) Explain why the above proof technique fails for $p = \infty$

(d) (2 points) Show $(\ell^\infty)^* \neq \ell^1$ (Hint: use the Hahn-Banach theorem)

Problem 9: (5 points) Dini's monotone convergence theorem: let $(f_n) \subset C([0, 1])$ be a monotone decreasing sequence that converges pointwise to $f \in C([0, 1])$. Prove f_n converges uniformly to f.

Problem 10: (5 points) Let X be a normed linear space and M a closed subspace, and let $x_0 \notin M$. Define

$$d = \operatorname{dist}(M, x_0) \stackrel{\text{def}}{=} \inf_{y \in M} \|y - x_0\|.$$

Prove there exists $\varphi \in X^*$ such that $\varphi(x_0) = 1$, $\varphi(y) = 0$ for all $y \in M$, and $\|\varphi\| = 1/d$.

Grading The grades were curved so that the percentage was raised to the .6 power, which is more beneficial to the lower scores than simply adding points or reducing the total possible points to the best score (which as 98/107).