Final exam Selected Solutions
APPM 5440 Fall 2014 Applied Analysis

Date: Tuesday, Dec. 15 2014, 10:30 AM to 1 PM
You may assume all vector spaces are over the real field unless otherwise specified.
Your proofs may use any major result discussed in class (if you are unsure, please ask).
Total points possible: 107. **N.B.** Unlike the homeworks, the grades may be curved.
Note that points are *not* distributed according to difficulty, e.g., you may wish to save problems 9 and 10 for last since they are not worth many points.

Problem 1: Compactness (12 points, 2 points each)

(a) State the *topological* definition of a **compact** set
(b) State the definition of a **sequentially compact** set
(c) State the simplified definition of a compact set in a finite-dimensional metric vector space
(d) State a definition of a **pre-compact** set
(e) State a definition of a **compact** linear operator on a normed vector space
(f) Give an example of a set that is bounded but not totally bounded

Solution: These are all in the book. (a) is just "every open cover has a finite subover" (not *countable* subcover; and make sure not to confuse this definition with the definition of a closed set in a metric space. (b) is "every sequence has a convergent subsequence", and note that we do *not* require the sequence to be bounded in the statement (that will be implicitly enforced, since every compact set is bounded). For (c), "closed and bounded" was the intended answer, but "complete and totally bounded" was accepted as well, since this is not false. Note that if we are over the real field, then all finite dimensional vector spaces are isomorphic to $\mathbb{R}^n$ for some n, hence "complete" is equivalent to "compact"; non-complete spaces, like $\mathbb{Q}^n$, are not isomorphic to $\mathbb{R}^n$ since these are vector spaces over $\mathbb{Q}$ but not over $\mathbb{R}$.

For (e), recall that a compact operator maps bounded sets to *pre*-compact sets, not to compact sets. I took off a point if your answer said that bounded sets were mapped to closed sets (or compact sets), since mapping closed sets to closed sets is an entirely different property.

For (f), the unit ball in any infinite dimensional space is an example (and any example must necessarily be in infinite dimensions, otherwise bounded implies totally bounded).

Problem 2: Major theorems (14 points, 2 points each) State the following theorems. You may choose *one* of these to skip (please indicate which one you do not want graded)

(a) State the Arzelà-Ascoli theorem
(b) State the Weierstrass Approximation theorem
(c) State the Banach contraction mapping theorem
(d) State the Hahn-Banach theorem (any version)
(e) State the open-mapping theorem
(f) State Grönwall's inequality
(g) State Bessel's inequality regarding an orthonormal set $U = (u_n)_{n\in\mathbb{N}}$ (you do not need to consider uncountable sets)
(h) State at least three equivalent definitions of an orthonormal basis (again, you do not need to consider uncountable sets)

Solution: See the book. For (b), I took off 1/2 point if you did not state that this only applied to functions over a compact interval of $\mathbb{R}$; it is *not* true over all of $\mathbb{R}$ (indeed, the L_∞ norm is not even a valid norm in this case, since it can be infinity).

For (c), I took off a full point if you did not require the space to be complete, since this is important (it need not be Banach; a complete metric space is OK, but I didn't penalize this).

Also note that a contraction is defined as a map $T : X \to X$ where

$$\exists c \in [0, 1) \text{ such that } \forall x \in X,\ \|Tx\| \le c\|x\|.$$

(here I am assuming T is linear and X is a Banach space, but similar definitions apply in general). This is *not the same* as a map T' where

$$\forall x \in X,\ \|T'x\| < \|x\|.$$

This is a subtle but important point! In the former case, we can derive that $\|T^n x\| \le c^n\|x\| \to 0$, but this conclusion does not apply in the latter case.

For (d), we need a linear subspace Y inside X (not a subset), and X must be a normed linear space. The subspace Y need not be closed. I didn't take off points if you missed just one of these little details. Also note that the extension need not be unique.

For (e), the statement is that if there is a linear bounded bijection between two Banach spaces, then the inverse is bounded. If the spaces are not complete, then the inverse need not be bounded, so the completeness of the spaces is crucial to the theorem and thus I took off a point if you missed this.

For (f), I took off a half-point if you forgot to require the non-negativity of the functions.

For (g), I took off a half-point if you didn't square the terms; the answer should be $\sum_n |\langle u_n, x\rangle|^2 \le \|x\|^2$ (many people also added the other 2 parts of the theorem in the book, but I didn't take off points if these were missing). Make sure that all your quantities are real numbers – think of this as a "dimensional analysis" check. E.g., you cannot compare $\sum\langle u_n, x\rangle \le \|x\|$ because $\langle u_n, x\rangle$ might be complex while $\|x\|$ is always real (even if the vector space is over the complex field).

For (h), make sure that you included the assumption that the set was orthonormal, and *then* made the additional requirements (e.g., totality, trivial orthogonal complement, maximal, Parseval's identity), otherwise it is not valid.

For reference, 4 students skipped the Bessel question, 3 skipped the open-mapping theorem, 1 skipped Grönwall's inequality, 2 skipped the Hahn-Banach theorem, 1 skipped the Weierstrass theorem, and 1 skipped the orthonormal basis question.

Problem 3: (16 points, 2 points each) Weak topologies and convergence. Let X be a normed linear space. You do not need to justify your answer.

(a) What does it mean to say a set $F \subset X$ is weakly closed?

(b) If F is weakly closed, is it necessarily also closed?

(c) If F is closed, is it necessarily also weakly closed?

(d) If $(x_n) \subset X$ and $x_n \rightharpoonup x$, then does $\|x_n\| \to \|x\|$?

(e) Are weak convergence and strong convergence the same if $X = \mathbb{R}$ (with the standard Euclidean norm)?

(f) Let X be a Banach space and X^* its topological dual. What is weak-* convergence?

(g) When is weak-* convergence equivalent to weak convergence?

(h) Let X and Y be Banach spaces. What does it mean to say that a sequence (T_n) in $\mathcal{B}(X, Y)$ *converges strongly* (*not* uniformly)? i.e., strong operator convergence.

Solution:

(a) It means $(x_n) \subset F$ and $x_n \rightharpoonup x$ implies $x \in F$.

(b) Yes, since to show F is closed, let $(x_n) \subset F$ and $x_n \to x$, then $x_n \rightharpoonup x$ as well, so $x \in F$, hence F is closed. Another way to see this is that the strong topology is stronger/finer than the weak topology, and consistent with it, which means all open sets in the weak topology are open in the strong topology but not vice-versa. Since closed sets are just the complement of open sets, the same conclusion holds with closed sets.

(c) No, since $D = \{x \in \ell^3 \mid \|x\| = 1\}$ is closed but not weakly closed, as we showed in our homework.

(d) No, since if (x_n) is orthonormal in ℓ^p for $p > 1$ then $x_n \rightharpoonup 0$ but $\|x_n\| = 1$ so $\lim_{n\to\infty} \|x_n\| = 1 \neq \|0\| = 0$.

(e) Yes. For example, $1 \in \mathbb{R}$ so $\langle 1, \cdot \rangle$ is a bounded linear functional. In fact, weak convergence and strong convergence are equivalent in any finite dimensional space. This can be proven using the facts from §5.4 from the book.

(f) Weak-* applies when a sequence in the dual, $(\varphi_n) \subset X^*$, converges in the sense that $\forall x \in X$, $\varphi_n(x) \to \varphi(x)$.

(g) When the space is reflexive ($X = X^{**}$). Some students said "in finite dimensions" which is not the answer I wanted, but this is true (since in finite dimensions, all spaces are reflexive). Also note that $X = X^*$ is another sufficient but not necessary condition (since this implies that $X = X^{**}$ of course).

(h) It means $\forall x \in X$, $\|T_n x - Tx\| \to 0$

Problem 4: (15 points) Let $(f_n) \subset C(I)$, and consider

$$\lim_{n\to\infty} \int_I f_n(x)\, dx \stackrel{?}{=} \int_I f(x)\, dx.$$

When is this necessarily true? If it is true, prove it, and if it is false, provide a counter-example.

(a) (5 points) f_n converges uniformly to f, and $I = [a, b]$.

(b) (5 points) f_n converges point-wise to f, and $I = [a, b]$.

(c) (5 points) f_n converges uniformly to f, and $I = \mathbb{R}$.

Solution:

(a) This follows since $\left|\int_I (f_n(x) - f(x))\, dx\right| \le \|f_n - f\|_\infty \cdot (b - a)$ (see also the homework solution in homework 4).

(b) See the homework solution (problem 2.2, in homework 4); this is not true and you can find counter-examples. You can even find counter-examples where f is continuous (e.g., $f = 0$).

(c) This is not true. Consider f_n representing a triangular region where the height shrinks and the base elongates as $n \to \infty$ in such a way that the area under the triangle is always 1. Then f_n converges to 0 uniformly. This was an example where all the integrals are finite, and f is even continuous, but the equality does not hold.

If you do not require finite integrals, then taking $f_n = 1/n$ is a very simple counter-example, since this converges to $f = 0$.

Note that this problem was worth a lot of points because I consider it fundamental.

Problem 5: (10 points) Prove Lemma 1.63: a sequentially compact metric space X is separable. You may use other relevant theorems in your proof.

Solution: See the book; use total boundedness (this is why I stated the hint about other theorems) and take a union of epsilon-nets. You should not use the sequential compact definition directly, since this is hard to generalize to *all* points in the set. In particular, note that there can be uncountably many sequences, so dealing with these is not easy.

Problem 6: (5 points) Let X be a space of points, and $\mathcal{T}_d$ be the discrete topology $\mathcal{P}(X)$, and $\mathcal{T}_t$ be the trivial topology $\{\emptyset, X\}$. Consider the identity map $\mathcal{I} : (X, \mathcal{T}_t) \to (X, \mathcal{T}_d)$. Is this map continuous? Briefly prove your answer. *Solution*: A map f is continuous iff the pre-image of an open set is open. Therefore, unless X consists of a single point, there is an open set in the $\mathcal{T}_d$ topology that is not in the $\mathcal{T}_t$ topology, and so the identity map cannot be continuous.

Problem 7: (10 points) Let X be a normed linear space. A series $\sum x_n$ in X is *absolutely convergent* if $\sum \|x_n\|$ converges to a finite value in $\mathbb{R}$.

(a) (7 points) Prove that if X is a Banach space, then every absolutely convergent series converges

(b) (3 points) Conversely, prove that if every absolutely convergent series converges, then X must be Banach.

Solution: This was exercise 1.20 in the book; see homework 3 solutions. The first part of this should not be too difficult which is why it was worth a lot of points; the second part of this problem is a bit tricky (one person received full credit).

Problem 8: (15 points) Show that for $1 \le p < \infty$, then $(\ell^p)^* = \ell^q$ (in the sense that the two spaces are isometrically isomorphic) for $1/q = 1 - 1/p$, and that $(\ell^\infty)^* \neq \ell^1$. You may take the field to be $\mathbb{R}$, and you may use standard theorems and inequalities that have been discussed in class. Specifically,

(a) (5 points) Show $(\ell^p)^* \subset \ell^q$ for $1 \le p < \infty$.

(b) (5 points) Show $\ell^q \subset (\ell^p)^*$ for $1 \le p < \infty$.

(c) (3 points) Explain why the above proof technique fails for $p = \infty$

(d) (2 points) Show $(\ell^\infty)^* \neq \ell^1$ (Hint: use the Hahn-Banach theorem)

Solution: Parts of this were a homework problem in homework 9; see the homework solutions. I consider this extremely fundamental, and it synthesizes several topics, so please study this problem well.

In part (a), you need to use sequential continuity of the inner product (not triangle inequality, not Hölder's inequality). See proof sketch below.

Part (b) is easy using Hölder's inequality, and the linearity is obvious. Make sure that the objective of what you want to show is clear; many students confused parts (a) and (b).

For part (c), it is not that Hölder's inequality fails, rather it is the partial sums do not converge in ℓ^∞ so you cannot use sequential continuity. Several answers used an example of $(1, 1, 1, \ldots)$ as a potential dual vector that is not in ℓ^1, but this does not lead to a *bounded* linear functional; also, the issue is not that some bounded linear functionals cannot be represented by a sequence in ℓ^1, but that they cannot be represented by a sequence at all!

For part (d), see the solutions in HW 9 (but note that another proof uses separability arguments); the idea is to look at the subspace of all sequences that have a limit.

For part (a), the proof sketch goes like this: let (e_n) be the unit basis elements, and for an arbitrary dual element φ, define $y = (y_n)$ with $y_n = \varphi(e_n)$, and define $\varphi_y(x) = \langle y, x\rangle$. Now, we claim two things: first, that $\varphi(x) = \varphi_y(x)$ for all x, and that $y \in \ell^\infty$.

For the first claim, we use the fact that $x^{(n)} \stackrel{\text{def}}{=} \sum_{k=1}^n x_k e_k$ converges to x in the ℓ^p norm as long as $p < \infty$. If $p = \infty$ this part fails. Now, use the sequential continuity of the inner product to see that

$$\varphi(x) = \varphi\left(\lim_{n\to\infty} x^{(n)}\right) = \lim_{n\to\infty} \varphi(x^{(n)}) = \lim_{n\to\infty} \langle y, x^{(n)}\rangle = \langle y, x\rangle$$

where we used the linearity of φ (we had to first reduce to $x^{(n)}$ since this has only finitely many terms, and the linearity only applies to finitely many terms). So we use the linearity of φ in an essential way.

For the second part, showing $y \in \ell^q$, we can do the following (note that no student answered this sufficiently). Let $y^{(n)} = \sum_{k=1}^n y_k e_k$. Our key tool is now the boundedness of φ. Without loss of generality, assume $\|\varphi\| = 1$. For all n, $\|y^{(n)}\|_q$ certainly exists since this is a finite sum. Now define $x_k = y_k^{q-1} = y_k^{q/p}$ since $1/p + 1/q = 1$. We also have $x = (x_k) \in \ell^p$ since this is a finite sum as well. We will assume wlog that we are over the real field and $y_k \geq 0$; if not, we could modify x_k to include a sign term as well. Now we have

$$\|y^{(n)}\|_q^q = \sum_{k=1}^n y_k y_k^{q-1} = \langle y^{(n)}, x \rangle = \varphi(x) \leq \|\varphi\| \|x\|_p = \left(\sum_{k=1}^n (y_k^{p/q})^p \right)^{1/p} = \|y^{(n)}\|_q^{q/p}$$

and $\|y^{(n)}\|^q \leq \|y^{(n)}\|^{q/p}$ for $p \geq 1$ implies that $\|y^{(n)}\| \leq 1$. Thus $\lim_{n\to\infty} \|y^{(n)}\|_q$ exists since it is an increasing and bounded sequence of real numbers, so $\|y\|_q$ is finite, so $y \in \ell^q$. Note that you cannot use corollaries of the Hahn-Banach for this part since that would require circular logic about the nature of the dual spaces.

Problem 9: (5 points) Dini's monotone convergence theorem: let $(f_n) \subset C([0,1])$ be a monotone decreasing sequence that converges pointwise to $f \in C([0,1])$. Prove f_n converges uniformly to f.

Solution: This was a homework problem (2.11); see the homework 5 solutions.

Problem 10: (5 points) Let X be a normed linear space and M a closed subspace, and let $x_0 \notin M$. Define

$$d = \operatorname{dist}(M, x_0) \overset{\text{def}}{=} \inf_{y \in M} \|y - x_0\|.$$

Prove there exists $\varphi \in X^*$ such that $\varphi(x_0) = 1$, $\varphi(y) = 0$ for all $y \in M$, and $\|\varphi\| = 1/d$.

Solution: We use the Hahn-Banach theorem. Let $\widetilde{M} = \operatorname{span}(M, x_0)$ and define $\psi(x) = d^{-1}\operatorname{dist}(M, x)$. On $\widetilde{M}$, this is a bounded linear functional. First, we show it is linear. We start by showing $\psi(\lambda x) = \lambda \psi(x)$. We have

$$\operatorname{dist}(M, \lambda x) = \inf_{y \in M} \|y - \lambda x\| = \inf_{\lambda y \in M} \lambda \|y - x\| = \lambda \operatorname{dist}(M, x)$$

since $y \in M \iff (\forall \lambda) \lambda y \in M$ since M is a linear subspace.

Now we show that $\varphi(x + x') = \varphi(x) + \varphi(x')$ if $x, x' \in \widetilde{M}$. We can write $x = m + \gamma x_0$ and $x' = m' + \gamma' x_0$ since they are in $\widetilde{M}$. Clearly $\varphi(x) = d^{-1}\gamma$ and $\varphi(x') = d^{-1}\gamma'$. Also, $x + x' = m'' + (\gamma + \gamma')x_0$ and we see $\varphi(x + x') = d^{-1}(\gamma + \gamma')$ since $m'' \overset{\text{def}}{=} m + m' \in M$.

Hence ψ is linear, so we now show it is bounded. Since $0 \in M$, $\inf_{y \in M} \|y - x\| \leq \|x\|$ for all x, hence ψ is bounded by d^{-1}. Furthermore $\|\psi\| = d^{-1}$; this does not follow immediately and is in fact a bit tricky. One way to show this is the following: for every $\varepsilon > 0$, define $y_\varepsilon \in M$ such that $\|y_\varepsilon - x_0\| < d + \varepsilon$. Then $x' = x - y_\varepsilon \in \widetilde{M}$ and $\operatorname{dist}(M, x') = \operatorname{dist}(M, x_0) = d$, and $\|x'\| < d + \varepsilon$. So

$$\psi(x') = d^{-1}d = 1 > d^{-1}(d + \varepsilon) > d^{-1}\|x'\|$$

therefore $\|\psi\| \geq d^{-1}$ since ε was arbitrary, so combined with $\|\psi\| \leq d^{-1}$ we have $\|\psi\| = d^{-1}$.

We also have $\psi(x_0) = 1$ and $\psi(m) = 0$ for $m \in M$, so we have satisfied the requirements.

Therefore we can apply the Hahn-Banach theorem to extend this functional to the entire space. This completes the proof.

Note: our explicit definition of ψ is not necessarily a bounded linear functional on *all* of X. Example: in $\mathbb{R}^3$, let M be the x-axis, and $x_0 = (0,1,0)$ and $w = (0,0,1)$, then with the Euclidean distance, $\operatorname{dist}(M, x_0) = \operatorname{dist}(M, w) = 1$ but $\operatorname{dist}(M, x + w) = \sqrt{2} \neq 2$, so this is not linear. Thus using the Hahn-Banach to extend from $\widetilde{M}$ to X is essential.

Note: we cannot use the projection theorem since this is not necessarily a Hilbert space. Your answer cannot use any type of orthogonality. Recall exercise 6.2 in the book, whose purpose was to prove that in a Banach space, the conclusions of the projection theorem may fail to hold.

Note: we needed M to be closed otherwise it is possible for $d = 0$ even if $x_0 \notin M$, and if that were true, it would not be possible for $\varphi(x_0) = 1$.

Grading The grades were curved so that the percentage was raised to the .6 power, which is more beneficial to the lower scores than simply adding points or reducing the total possible points to the best score (which as 98/107).