Final Exam
APPM 5450 Spring 2015 Applied Analysis 2

Date: Thursday, May. 7 2015
Instructor: Dr. Becker   Your name: ____________________

If the mathematical field is not specified, you may assume it is $\mathbb{R}$ or $\mathbb{C}$ at your convenience. The symbol $\mathcal{H}$ denotes an arbitrary Hilbert space. Your proofs may use any major result discussed in class (if you are unsure, please ask).

Total points possible: 104.

N.B. Unlike the homeworks, the grades may be curved. Points are not distributed according to difficulty. When justifying your answer, do as little work as possible, e.g., avoid explicitly evaluating integrals if not necessary. Partial credit is possible.

For problems 1 and 2, PLEASE WRITE DIRECTLY ON THIS SHEET

Problem 1: (20 pts) Definitions. State the following definitions and/or theorems. You may skip one definition (please clearly mark which one should not be graded). 2 points each.

(1) Define the Sobolev space $H^s(\mathbb{T})$ for $s > 0$.

(2) Sobolev embedding theorem, any version

(3) Banach-Alouglu theorem, any variant

(4) Briefly define the Fourier transform on $\mathcal{S}(\mathbb{R})$ (the space of Schwartz functions).

(5) Briefly define the Fourier transform on $\mathcal{S}(\mathbb{R})^*$.

(6) Briefly define the Fourier transform on $L^2(\mathbb{R})$.

(7) What does it mean for $(\varphi_n) \subset \mathcal{S}$ to converge to a limit φ?

(8) Lebesgue dominated convergence theorem

(9) Fubini's theorem

(10) Define what it means for a function $f : X \to Y$ to be measurable with respect to measure spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$.

(11) In the above definition, if $Y = \overline{\mathbb{R}}$ and $\mathcal{B}$ is the Borel σ-algebra generated by the standard topology on $\overline{\mathbb{R}}$, what is a simplified definition of a measurable function?

Problem 2: (30 pts) Mark true/false (or yes/no). No justification needed. $\mathcal{H}$ denotes a Hilbert space. 2 points each.

(1) Let $C, D \subset \mathcal{H}$. If $C = D^\perp$, is $C^\perp = D$? ______

(2) If $\mu(X) < \infty$, then $L^p(X) \subset L^q(X)$ if $p \leq q$. ______

(3) If $\mu(X) < \infty$, then $L^p(X) \subset L^q(X)$ if $p \geq q$. ______

(4) If $\mu(X) = \infty$, then $L^p(X) \subset L^q(X)$ if $p \leq q$. ______

(5) If $\mu(X) = \infty$, then $L^p(X) \subset L^q(X)$ if $p \geq q$. ______

(6) Is the Heaviside function $H(x) = \chi_{(0,\infty)}(x)$ weakly differentiable? ______

(7) Is it possible that in $\ell^1(\mathbb{N})$, weak convergence always implies strong convergence? ______

(8) $L^p(\mathbb{R})$ is separable for all $1 \leq p \leq \infty$. ______

(9) $L^p([0,1])$ is separable for all $1 \leq p \leq \infty$. ______

(10) $C([0,1])$ is dense in $L^\infty([0,1])$. ______

(11) $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. ______

(12) $C_c(\mathbb{R})$ is complete with respect to the uniform norm. ______

(13) Let $f \in L^2(\mathbb{T})$ and define the partial sum $f_N = \sum_{n=-N}^{N} \widehat{f}_n e_n$ where (e_n) is the Fourier basis. Does $\|f - f_n\|_{L^2} \to 0$? ______

(14) If $f, g \in L^1(\mathbb{R})$, is $fg \in L^1(\mathbb{R})$? ______

(15) If $f, g \in L^1(\mathbb{R})$, is $f * g \in L^1(\mathbb{R})$? ______

Problem 3: (12 pts) Short response. 2 points each. For examples of functions, don't forget to specify their domain.

(1) The space $L^2(\mathbb{T})$ contains periodic functions, but the functions are not defined pointwise because they are really equivalence classes. If they are not defined pointwise, how can they be periodic? Briefly discuss.

(2) On $I = [-\pi, \pi]$, $f(x) = x$ is very smooth, i.e., $f \in C^\infty(I)$, whereas $g(x) = |x|$ is not a smooth, i.e., $g \in C(I) \setminus C^1(I)$. Do you expect the Fourier coefficients of f to decrease faster than those of g? Is this true?

(3) Give an example of an operator that is positive but not coercive.

(4) Give an example of functions (f_n) that converge pointwise a.e. to f but $\lim_{n\to\infty} \int f_n \neq \int f$.

(5) Give an example of a function that is Lebesgue integrable but not Riemann integrable.

(6) Give an example of a function that has an improper Riemann integral but is not Lebesgue integrable.

Problem 4: (6 pts) Convergence. 3 points each.

(1) Let $f_n(x) = e^{in\pi x} \in L^2([0,1])$. Does f_n converge strongly, or weakly, and if so, what is the limit? Justify your answer.

(2) Let $f_n(x) = e^{in\pi x}\chi_{[-n,n]} \in L^2(\mathbb{R})$. Does f_n converge strongly, or weakly, and if so, what is the limit? Justify your answer.

Problem 5: (6 pts) Convergence and Integrals. Let (f_n) and f be integrable functions on $[1,\infty)$ such that $f_n \to f$ a.e. Give a short proof or a counter-example for the following statements: (3 points each)

(1) If $f_n \to f$ uniformly, then $\lim_{n\to\infty} \int_1^\infty f_n = \int_1^\infty f$.

(2) If (f_n) is monotone decreasing, then $\liminf_{n\to\infty} \int_1^\infty f_n = \int_1^\infty f$.

Problem 6: (6pts) Spectral theory. Let $A \in \mathcal{B}(\mathcal{H})$. 3 points each.

(1) Prove $\lambda \in \sigma_r(A)$ implies $\overline{\lambda} \in \sigma_p(A^*)$.

(2) If $A = A^*$, prove $\sigma_r(A) = \emptyset$.

Problem 7: (8 pts) Fourier transform. 2 points each.

(1) Let $f(x) = e^{i\omega x}$ for $\omega \in \mathbb{R}$ be a function on $\mathbb{R}$. Which of the following spaces does f live in: $\mathcal{S}(\mathbb{R}), \mathcal{S}^*(\mathbb{R}), L^1(\mathbb{R}), L^2(\mathbb{R})$ (combinations allowed, e.g., "none" or "all")?

(2) Let $\mathcal{F}$ be the Fourier transform. What is $\widehat{f} \stackrel{\text{def}}{=} \mathcal{F}(f)$ for f as above?

(3) Let $\varphi \in \mathcal{S}(\mathbb{R})$. What is $\langle \delta', \varphi \rangle$?

(4) Let $\varphi \in \mathcal{S}(\mathbb{R})$. What is $\delta' * \varphi$?

Problem 8: (3 pts) Given an example of a measurable space (X, μ) and measurable sets (E_n) such that $E_1 \supseteq E_2 \supseteq E_3 \dots$ and

$$\lim_{n\to\infty} \mu(E_n) \neq \mu\left(\bigcap_n E_n\right).$$

Problem 9: (13 pts) Bounded linear operators.

(1)* (1 pt) Let X be a Banach space. If $\varphi(x) = \varphi(y)$ for all $\varphi \in X^*$, prove $x = y$.

(2) (2 pts) A *closed* operator $T : X \to Y$, X and Y normed linear spaces, is such that if for $(x_n) \subset X$, $x_n \to x$ and $T(x_n) \to y$, then $T(x) = y$. Explain how this differs from a *continuous* operator, and state whether closed operators are continuous, or vice-versa, or neither.

(3) (0 pts, fact) Let $\mathcal{H}$ be a Hilbert space, and let $A : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{H} \to \mathcal{H}$ be operators (not necessarily linear nor bounded) with the property that $\langle Ax, y\rangle = \langle x, By\rangle$ for all $x, y \in \mathcal{H}$. Then you can prove A (and hence B) must be linear operators.

(4) (3 pts) Under the same assumptions as part (3), prove A (and hence B) must be bounded as well. Hint: you may use the following corollary of the open-mapping theorem known as the "Closed Graph Theorem": if X and Y are Banach and $A : X \to Y$ is linear, then A is closed iff it is bounded.

(5) (4 pts) Let $H^1(\mathbb{T})$ be the Sobolev space on the torus. 2 points each:

i. Define the weak derivative (denote the operator by D).

ii. Is $D : H^1(\mathbb{T}) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$ bounded? Briefly justify

(6) (3 pts) In chapter 10, which we did not cover, the book defines a concept called the "formal adjoint" of the weak derivative D. Can this be the same concept of "adjoint" that we discussed this semester? Please discuss why or why not.