Final Exam
APPM 5450 Spring 2016 Applied Analysis 2

Date: Wed, May. 4 2016
Instructor: Dr. Becker Your name: ______________________

If the mathematical field is not specified, you may assume it is $\mathbb{R}$ or $\mathbb{C}$ at your convenience. The symbol $\mathcal{H}$ denotes an arbitrary Hilbert space. Your proofs may use any major result discussed in class (if you are unsure, please ask). Spend your time on the problems worth a lot of points: problems worth more points are not necessarily harder. Partial credit is possible on all problems except the True/False.

Total points possible: 100.

For problems 1 and 2, PLEASE WRITE DIRECTLY ON THIS SHEET

Problem 1: (22 pts) Definitions and short answer, 2 points each.

(1) Define the Sobolev space $H^s(\mathbb{T})$ for $s > 0$.

(2) State the Banach-Alouglu theorem, any variant

(3) What does it mean for $(\varphi_n) \subset \mathcal{S}$ to converge to a limit φ?

(4) What does it mean for $(T_n) \subset \mathcal{S}^*$ to converge to a limit T?

(5) State Fatou's lemma

(6) Let X and Y be normed linear spaces, and $T : X \to Y$ linear. Define what it means for T be compact.

(7) How can we make sense of $g(t) = \int_{\mathbb{R}} 1 \cdot e^{i\omega t}\, d\omega$?

(8) Let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ and $g : (Y, \mathcal{B}) \to (Z, \mathcal{C})$ be measurable functions, and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be σ-algebras on the spaces X, Y, Z respectively. Is $g \circ f : (X, \mathcal{A}) \to (Z, \mathcal{C})$ measurable? Briefly justify why it is, or give a counter-example why it is not.

(9) Let X be a linear space and $P : X \to X$ a projection. Is $\text{ran}(P)$ necessarily closed? When $\text{ran}(P)$ is closed, is P a bounded operator? (Prove or disprove).

(10) Give an example of a linear operator $T \in \mathcal{B}(X, Y)$ for Banach spaces X and Y that does not have closed range.

(11) Let $X = L^{4/3}([0, 1])$. Is the set $D = \{f \in X : \|f\| = 1\}$ weakly closed? Briefly justify or provide a counterexample.

Problem 2: (32 pts) Mark true/false (or yes/no). No justification needed. 2 points each.

(1) Let $(f_n) \subset H^1$ where $H^1 \subset L^2(\mathbb{R})$ is a Sobolev space.
If there is $f \in L^2$ such that $\lim_{n\to\infty} \|f_n - f\|_{L^2} = 0$, is $f \in H^1$? . . . . ______

(2) Let ∂ be the differential operator on L^2 that maps $f \in H^1$ to its weak derivative ∂f
(so ∂ is not defined on all of L^2). Is ∂ a bounded linear operator with respect to $L^2(\mathbb{R})$? . . . ______

(3) If a linear operator is compact, then it is also bounded. . . . . ______

(4) Let X be a normed linear space and $T : X \to X$ a linear operator (not necessarily bounded),
and let $x = \sum_{n=1}^{\infty} \alpha_n x_n \in X$ for $x_n \in X$ and scalars α_n. Then $T(x) = \sum_{n=1}^{\infty} \alpha_n T(x_n)$. . . . . ______

(5) Let X be a normed linear space. If X is reflexive, then it must be Banach. . . . . ______

(6) If $\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$ exists in the Riemann sense, then f is Lebesgue integrable. . . . . ______

(7) All subspaces are closed. . . . . ______

(8) In a separable Banach space, a Schauder basis is a set such that every element of the
Banach space can be written as a finite linear combination of basis elements. . . . . ______

(9) Every Hilbert space has an orthonormal basis. . . . . ______

(10) The right-shift operator $\mathcal{S}$ on $\ell^\infty(\mathbb{N})$ is onto. . . . . ______

(11) Let X be a normed linear space. We say a sequence $(\varphi_n) \subset X^*$ converges to φ in the
weak* sense if it converges weakly with respect to X^{**}, i.e., $\forall f \in X^{**}$, $f(\varphi_n) \to f(\varphi)$. . . . . ______

(12) Let X, Y be normed linear spaces and $X \subset Y$. If $(x_n) \subset X$ converges weakly with respect
to X, does it also converge weakly with respect to Y? . . . . ______

(13) If P is an orthogonal projection on a Hilbert space $\mathcal{H}$ then $\mathcal{H} = \operatorname{ran}(P) \oplus \ker(P)$. . . . . ______

(14) Let H be the heaviside function $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ if $x < 0$.
Then the regular distribution T_H has a weak derivative. . . . . ______

(15) Define $g(t) = \int_{-1}^{1} s^3 e^{ist}\,ds$. Is $g \in L^1(\mathbb{R})$? . . . . ______

(16) (For the same g as above). Is $g \in L^2(\mathbb{R})$? . . . . ______

Problem 3: (5 pts). Let $(e_n) \subset \mathcal{H}$ be any orthonormal set. Prove $e_n \rightharpoonup 0$.

Problem 4: (5 pts). Let $A \in \mathcal{B}(\mathcal{H})$ be a compact self-adjoint operator. Prove that for all $R > 0$, there are only finitely many eigenvalues with magnitude greater than R.

Problem 5: (10 pts). Let $f \in L^2(\mathbb{R})$ and $\mathcal{F}$ represent the Fourier transform on $L^2(\mathbb{R})$.

(1) (5 pts). How is $\mathcal{F}$ defined?

(2) (5 pts). If f is non-negative, prove

$$\big(\mathcal{F}f\big)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x}\, d\lambda$$

is well-defined (using the Lebesgue measure λ)

Problem 6: (5 pts). Let X be a normed vector space, and $x_n \rightharpoonup x$. Prove $(\|x_n\|)$ is bounded.

Problem 7: (10 pts). Cantor set. Define $F_1 = [0, 1/3] \cup [2/3, 1]$, and then remove the middle third of each interval to define $F_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8, 9, 1]$, and remove the middle third of each interval of F_2 to define F_3, and construct all F_n in this recursive fashion. Then the closed set $F \stackrel{\text{def}}{=} \cap_{n=1}^{\infty} F_n$ is called the Cantor set. A number $x \in [0, 1]$ belongs in F if and only if it has a base three expansion (which may not be unique) that contains no $1's$, e.g., $1/3 = 0.1\bar{0}$ in base 3, but we can also write $1/3 = 0.02\bar{2}$, hence $1/3 \in F$.

(1) (7 pts) Prove that the Lebesgue measure of F is zero, i.e., $\lambda(F) = 0$.

(2) (3 pts) Prove F is uncountable, and hence there exist uncountable sets with zero Lebesgue measure.

Problem 8: (10 pts). Let $0 \neq g \in L^p(\mathbb{R})$ for $1 < p < \infty$ be a fixed function, and $f_n(x) \stackrel{\text{def}}{=} g(x)\sin(n\pi x)$.

(1) (7 pts) Prove $f_n \rightharpoonup 0$.

(2) (3 pts) Prove f_n does not converge strongly. (Remark: g is arbitrary — you cannot choose it. For example, g may be the indicator function of the real numbers on $[0, 1]$.)

Problem 9: (1 pt) Prove that for $1 < p < \infty$, the dual of $\ell^p(\mathbb{N})$ is $\ell^q(\mathbb{N})$ where $p^{-1} + q^{-1} = 1$.