Final Exam Selected Solutions
APPM 5450 Spring 2016 Applied Analysis 2

Date: Wed, May. 4 2016
Instructor: Dr. Becker   Your name: ____________________

If the mathematical field is not specified, you may assume it is $\mathbb{R}$ or $\mathbb{C}$ at your convenience. The symbol $\mathcal{H}$ denotes an arbitrary Hilbert space. Your proofs may use any major result discussed in class (if you are unsure, please ask). Spend your time on the problems worth a lot of points: problems worth more points are not necessarily harder. Partial credit is possible on all problems except the True/False.

Total points possible: 100.

For problems 1 and 2, PLEASE WRITE DIRECTLY ON THIS SHEET

Problem 1: (22 pts) Definitions and short answer, 2 points each.

(1) Define the Sobolev space $H^s(\mathbb{T})$ for $s > 0$.

Solution: $\{f = 1/\sqrt{2\pi}\sum_{n\in\mathbb{N}} \hat{f}_n e^{inx} \in L^2(\mathbb{T}) \mid \sum_{n\in\mathbb{N}} n^{2s}|\hat{f}_n|^2 < \infty\}$. A response such as "functions with s-weak derivatives" did not get full credit as we need a definition like this to make sense of a non-integer weak derivative.

(2) State the Banach-Alouglu theorem, any variant

Solution: The unit ball is either (1) weakly compact, in a Hilbert space or reflexive Banach space, or (2) weak-* compact in a Banach space. The theorem does not apply to a general normed linear space — it must be at least Banach.

(3) What does it mean for $(\varphi_n) \subset \mathcal{S}$ to converge to a limit φ?

Solution: It means for all α, β multi-indices, then $\|\varphi_n - \varphi\|_{\alpha,\beta} \to 0$. The pseudo-norm is defined in eq. (11.3).

(4) What does it mean for $(T_n) \subset \mathcal{S}^*$ to converge to a limit T?

Solution: This means weak-* convergence, i.e., for all $\varphi \in \mathcal{S}$, $\langle T_n, \varphi\rangle \to \langle T, \varphi\rangle$.

(5) State Fatou's lemma

Solution: If $f_n \geq 0$ are measurable functions, then $\int \liminf f_n \leq \liminf \int f_n$. A good number of students forgot to require $f_n \geq 0$ (about a quarter of students did not get full credit).

(6) Let X and Y be normed linear spaces, and $T : X \to Y$ linear. Define what it means for T be compact.

Solution: Maps bounded sets to precompact sets is one possible answer; you lost points if you said mapped bounded sets to *compact* sets, since the operator need not map closed sets to closed sets (unless, e.g., it has a continuous inverse).

(7) How can we make sense of $g(t) = \int_{\mathbb{R}} 1 \cdot e^{i\omega t}\, d\omega$?

Solution: We can think of 1 as in $\mathcal{S}^*$, and then g is its Fourier transform, so also in $\mathcal{S}^*$, and in fact it is just a (scaled) delta function. Full credit awarded if you mentioned δ or "in the distributional sense" or similar.

(8) Let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ and $g : (Y, \mathcal{B}) \to (Z, \mathcal{C})$ be measurable functions, and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be σ-algebras on the spaces X, Y, Z respectively. Is $g \circ f : (X, \mathcal{A}) \to (Z, \mathcal{C})$ measurable? Briefly justify why it is, or give a counter-example why it is not.

Solution: True, straightforward.

(9) Let X be a linear space and $P : X \to X$ a projection. Is $\text{ran}(P)$ necessarily closed? When $\text{ran}(P)$ is closed, is P a bounded operator? (Prove or disprove).

Solution: (1) $\text{ran}(P)$ is not necessarily closed (but the range is always a subspace – so your answer should be consistent with your answer about closed subspaces in the T/F section).

(2) Yes, if range is closed, it is bounded, since the range is $\{x : x = Px\}$, so if $x_n \to x$ then $Px_n = x_n \to x$ and since the range is closed, $Px_n \to Py$ for some y, hence $x = y$ and $Px_n \to Px$, meaning P is sequentially continuous and hence bounded. (In fact, P is bounded if and only if its range is closed).

Many students assumed that if the range is closed, then P is orthogonal. This is not true; it is true that P orthogonal means the range is closed; and it is true that given a closed subspace M, you can define a unique orthogonal projection that has range M, but you could also define many non-unique non-orthogonal with range M.

More than half the class did poorly on this question! See Fig. 1 for a brief sketch of an example non-orthogonal projection. We have the basic results: a projection is bounded iff its range is closed; and the norm is 1 (or 0) iff the projection is orthogonal (Problem 3 from HW 4).

(10) Give an example of a linear operator $T \in \mathcal{B}(X, Y)$ for Banach spaces X and Y that does not have closed range.

Solution: We need to violate that $\|Tx\| \geq c\|x\|$ for all x. Take $X = Y = \ell^2(\mathbb{N})$ and define $T(x) = (x_1, 1/2x_2, 1/3x_3, \ldots)$ so then it is clear that T^{-1} is not a bounded linear operator; if the range were closed, then it would be complete, and the open mapping theorem would imply that T^{-1} actually is bounded.

Another way to think of it is to find any T such that $0 \in \sigma_c(T)$.

Another good example is $\big(Tf\big)(x) = xf(x)$ on $L^2([0,1])$, and we know from class that $\sigma_c(T) = [0,1]$.

About half the students did poorly on this question.

(11) Let $X = L^{4/3}([0,1])$. Is the set $D = \{f \in X : \|f\| = 1\}$ weakly closed? Briefly justify or provide a counterexample.

Solution: False. For example, take any orthonormal basis $(e_n) \subset L^2([0,1]) \subset L^{4/3}([0,1])$ (the L^p spaces are nested on compact intervals), then with respect to L^2, $e_n \rightharpoonup 0$, and because $(L^{4/3})^* = L^4 \subset L^2 = (L^2)^*$, it follows $e_n \rightharpoonup 0$ with respect to $L^{4/3}$ as well, hence we have a sequence $(e_n) \subset D$ but the weak limit is not in D, so D is not weakly closed.

More than half the class did poorly on this question. It was not intended to be a trick question (we used the notation D for $\|f\| = 1$, while we typically use B for $\|f\| \leq 1$). The Banach-Alouglu theorem applies to the closed unit ball B, not its boundary D. See problems 3 and 4 from Homework 1.

Problem 2: (32 pts) Mark true/false (or yes/no). No justification needed. 2 points each.

More than a quarter of the class missed numbers 1, 7, 8, 11, 12, 14–16; no one missed 3 and 6; and the rest were missed by at least two people.

(1) Let $(f_n) \subset H^1$ where $H^1 \subset L^2(\mathbb{R})$ is a Sobolev space.
If there is $f \in L^2$ such that $\lim_{n\to\infty} \|f_n - f\|_{L^2} = 0$, is $f \in H^1$? *Solution*: **False**. H^1 is complete using its own norm, but not with the L^2 norm.

(2) Let ∂ be the differential operator on L^2 that maps $f \in H^1$ to its weak derivative ∂f (so ∂ is not defined on all of L^2). Is ∂ a bounded linear operator with respect to $L^2(\mathbb{R})$? *Solution*: **False**, it is linear but not bounded, i.e., not sequentially continuous. H^1 is dense in L^2 (wrt L^2 norm), so if $f_n \to f \in L^2 \setminus H^1$ where $(f_n) \subset H^1$, then if ∂ were sequentially continuous, it means in fact f does have a weak derivative. It *is* continuous on $\mathcal{S}$ with the usual topology.

(3) If a linear operator is compact, then it is also bounded. *Solution*: **True**, since it maps bounded sets to precompact sets, and precompact sets are necessarily bounded; any linear operator that maps bounded sets to bounded sets has a bounded operator norm.

(4) Let X be a normed linear space and $T : X \to X$ a linear operator (not necessarily bounded), and let $x = \sum_{n=1}^{\infty} \alpha_n x_n \in X$ for $x_n \in X$ and scalars α_n. Then $T(x) = \sum_{n=1}^{\infty} \alpha_n T(x_n)$. *Solution*: **False**. Because this is an infinite sum, we need to use sequential continuity, not linearity, and we do not know T is sequential continuous unless it is bounded. This is true in finite dimensions, though, since all linear operators are bounded then.

(5) Let X be a normed linear space. If X is reflexive, then it must be Banach. *Solution*: **True**; we are saying $X = X^{**}$ and any dual space is always complete.

(6) If $\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$ exists in the Riemann sense, then f is Lebesgue integrable. *Solution*: **False**. If f is Riemann integrable, it is Lebesgue integrable, but this is not true for an *improper* Riemann integral such as the one listed above.

(7) All subspaces are closed. *Solution*: **False**, though true in finite dimensions. For example, $H^1 \subset L^2$ is not closed (cf. T/F question 1 above); or, the set of all polynomials is clearly a subspace, but it is not closed (since, e.g., on $[0,1]$, its closure under the sup-norm is $C([0,1])$ via Weierstrass). Note that the range of a linear operator is always a subspace, and not all linear operators have closed range.

(8) In a separable Banach space, a Schauder basis is a set such that every element of the Banach space can be written as a finite linear combination of basis elements. *Solution*: **False**, this is the definition of a Hamel basis (and it holds regardless of completeness or separability of the space); a Schauder basis relaxes the assumption that we have a *finite* linear combination. A lot of students missed this question.

(9) Every Hilbert space has an orthonormal basis. *Solution*: **True** (though it requires the axiom of choice, so a False answer would be acceptable if you wrote that you do not believe the axiom of choice).

(10) The right-shift operator $\mathcal{S}$ on $\ell^\infty(\mathbb{N})$ is onto. *Solution*: **False**.

(11) Let X be a normed linear space. We say a sequence $(\varphi_n) \subset X^*$ converges to φ in the weak* sense if it converges weakly with respect to X^{**}, i.e., $\forall f \in X^{**}$, $f(\varphi_n) \to f(\varphi)$. *Solution*: **False**, this is just weak convergence in X^* (there is no special name for it). To have weak* convergence, we would reduce the condition to just those $f \in X^{**}$ which can be defined by $f(\varphi) = \varphi(x)$ for some $x \in X$, so it is equivalent only if X is reflexive.

(12) Let X, Y be normed linear spaces and $X \subset Y$. If $(x_n) \subset X$ converges weakly with respect to X, does it also converge weakly with respect to Y? *Solution*: **True**, since $X \subset Y$, then $Y^* \subset X^*$.

(13) If P is an orthogonal projection on a Hilbert space $\mathcal{H}$ then $\mathcal{H} = \operatorname{ran}(P) \oplus \ker(P)$. *Solution*: **True**, since the range is closed and $P = P^*$ so it follows from $\mathcal{H} = \overline{\operatorname{ran}(P)} \oplus \ker(P^*)$.

(14) Let H be the heaviside function $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ if $x < 0$. Then the regular distribution T_H has a weak derivative. *Solution*: **False**. It has a distributional derivative (the delta function) but this is singular. A weak derivative would imply H is continuous (via Sobolev embedding) which it isn't. A weak derivative can be thought of as a special case of a distributional derivative (i.e., if we have a regular distributional derivative).

(15) Define $g(t) = \int_{-1}^{1} s^3 e^{ist}\,ds$. Is $g \in L^1(\mathbb{R})$? *Solution*: **False**. Think of g as the Fourier transform of the function $\hat{g}(s) = s^3 \chi_{[-1,1]}$, so if g were in L^1, we would need $\hat{g}$ to be continuous (Riemann-Lebesgue), which it is not due to the indicator function. Note: using a computer to calculate the integral, the leading order t term looks like $\cos(at)/t$, which supports our reasoning.

(16) (For the same g as above). Is $g \in L^2(\mathbb{R})$? *Solution*: **True**. Using the same $\hat{g}$ as above, we see $\hat{g} \in L^2(\mathbb{R})$, and the Fourier transform maps L^2 to L^2, so therefore $g \in L^2$. Almost all students missed this problem.

Problem 3: (5 pts). **Subject: Weak convergence, Bessel's** Let $(e_n) \subset \mathcal{H}$ be any orthonormal set. Prove $e_n \rightharpoonup 0$.

Solution: This follows immediately from Bessel's inequality. Note that we do not require the set to be a basis, i.e., it need not be total (you could extend it to be, but that is unnecessary).

Problem 4: (5 pts). **Subject: Spectrum, spectral theorem** Let $A \in \mathcal{B}(\mathcal{H})$ be a compact self-adjoint operator. Prove that for all $R > 0$, there are only finitely many eigenvalues with magnitude greater than R.

Solution: We know all eigenvalues are bounded in magnitude by $\|A\|$. If there were infinitely many eigenvalues of magnitude greater than R, then because the set $\{\lambda \mid R < |\lambda| \leq \|A\|\} \subset \mathbb{R}$ is pre-compact, any infinite sequence must have a convergent subsequence. Since all accumulation points of eigenvalues must be 0, by the spectral theorem, this is impossible. For full credit, you needed some kind of basic argument that an infinite sequence in a bounded subset of $\mathbb{R}$ contains a limit point (the quickest such argument being simply "bounded" = "pre-compact" in finite dimensions, i.e., Bolzano–Weierstrass). Partial credit for stating the spectral theorem correctly.

Another proof: if there were infinitely many eigenvalues λ_i with magnitude greater than R, then let (e_i) be a sequence of normalized eigenvectors corresponding to these eigenvalues, and the e_i are orthonormal since A is self-adjoint. By the previous problem, $e_i \rightharpoonup 0$, and then $Ae_i = \lambda_i e_i$ so $\|Ae_i\| \geq R$ so it cannot converge to 0, so A cannot be compact (compact means maps weakly convergent to convergent).

Problem 5: (10 pts). **Subject: Fourier transform, density, MCT** Let $f \in L^2(\mathbb{R})$ and $\mathcal{F}$ represent the Fourier transform on $L^2(\mathbb{R})$.

(1) (5 pts). How is $\mathcal{F}$ defined?

Solution: Via the density of $L^1 \cap L^2 \subset L^2$ (or $\mathcal{S} \subset L^2$), and using that L^2 is complete, we can apply the BLT theorem and therefore write $\mathcal{F}(f)$ as the limit of $\mathcal{F}(f_n)$ for any $(f_n) \subset L^1 \cap L^2$ with $\|f_n - f\|_2 \to 0$. Such a f_n could be defined as $f_n(x) = \begin{cases} f(x) & |x| < n \\ 0 & |x| \geq n \end{cases}$ thus we could write

$$\left(\mathcal{F}f\right)(\omega) = \frac{1}{2\pi} \lim_{n\to\infty} \int_{-n}^{n} f(x) e^{-i\omega x}\, dx. \tag{1}$$

(2) (5 pts). If f is non-negative, prove

$$\big(\mathcal{F}f\big)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x}\, d\lambda$$

is well-defined (using the Lebesgue measure λ)

Solution: This is a mistake, and in fact is not true (take $f(x) = \begin{cases} 1/x & |x| > 1 \\ 0 & \text{else} \end{cases}$ so that $f \in L^2 \setminus L^1$, then for $\omega = 1$, we basically have the integral of sinc, which is not integrable). We cannot just apply the MCT to (1) because even though $f \geq 0$ and f_n converges to f monotonically, it is not true that $f_n(x)e^{-i\omega x} \geq 0$ or that $f_n(x)e^{-i\omega x}$ converges monotonically.

Problem 6: (5 pts). **Subject: Weak convergence, uniform boundedness theorem, dual spaces** Let X be a normed vector space, and $x_n \rightharpoonup x$. Prove $(\|x_n\|)$ is bounded.

Solution: The basic idea is the uniform boundedness theorem (aka Banach-Steinhaus): if (φ_n) are bounded linear functionals on a Banach space, and if $\varphi_n(y)$ is bounded for each y (the bound depending on y perhaps), then in fact $\|\varphi_n\|$ is bounded.

To apply this, we have two issues: our x_n is not a functional, and we are not in a Banach space. Both problems are solved by mapping $x_n \mapsto \varphi_n \in X^{**}$ where $\varphi_n : X^* \to \mathbb{R}$ is defined as $\varphi_n(f) = f(x_n)$ (i.e., the canonical embedding). Using the appropriate norms, we also have $\|x_n\| = \|\varphi_n\|$. Now we operate on X^*, which is always Banach.

For any $f \in X^*$, $\varphi_n(f) = f(x_n)$ and this is a bounded sequence since $f(x_n)$ is a convergent sequence in $\mathbb{R}$. The result follows now from the uniform boundedness theorem applied to φ_n.

Comments: many students wanted to use Banach-Steinhaus but were concerned about the fact that X was not a Banach space. This does not matter. The proof of proposition 8.40 part (a) still works. It is also tempting to use Hahn-Banach to show the existence of a functional φ such that $\|\varphi\| = 1$ and, for a particular x, $\varphi(x) = \|x\|$. But this does not mean $\varphi(x - x_n) = \|x - x_n\|$, so this approach doesn't work (this was a very common error for students this year and last year — make sure you understand the issue).

Problem 7: (10 pts). **Subject: Measure theory, classical analysis** Cantor set. Define $F_1 = [0, 1/3] \cup [2/3, 1]$, and then remove the middle third of each interval to define $F_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8, 9, 1]$, and remove the middle third of each interval of F_2 to define F_3, and construct all F_n in this recursive fashion. Then the closed set $F \stackrel{\text{def}}{=} \cap_{n=1}^{\infty} F_n$ is called the Cantor set. A number $x \in [0, 1]$ belongs in F if and only if it has a base three expansion (which may not be unique) that contains no $1's$, e.g., $1/3 = 0.1\bar{0}$ in base 3, but we can also write $1/3 = 0.02\bar{2}$, hence $1/3 \in F$.

(1) (7 pts) Prove that the Lebesgue measure of F is zero, i.e., $\lambda(F) = 0$.

Solution: Each F_n is a finite union of intervals, so we can explicitly calculate its Lebesgue measure. Furthermore, we have $\lambda(F_1) < \infty$ and $F_{n+1} \subset F_n$, so applying the homework problem about continuity of measure,

$$\lambda(F) = \lambda(\cap F_n) = \lim_{n \to \infty} \lambda(F_n) = 0$$

since $\lambda(F_n) = (2/3)^n$.

Alternatively, you could make them into disjoint intervals, which is basically replicating the proof of the homework problem.

Finally, another quick proof is that since $F \subset F_n$ for all n, then $0 \le \lambda(F) \le \lambda(F_n) = (2/3)^n$ for all n hence $\lambda(F) = 0$.

(2) (3 pts) Prove F is uncountable, and hence there exist uncountable sets with zero Lebesgue measure.

Solution: We follow the classic Cantor diagonal proof that the real numbers are uncountable. Let $(x_n)_{n \in \mathbb{N}}$ be a proposed enumeration of F, and write out each x_n in its base 3 representation, e.g.,

$$\begin{aligned} x^1 &= 0.\textcircled{0}200202220\ldots \\ x^2 &= 0.2\textcircled{2}00200200\ldots \\ x^3 &= 0.02\textcircled{0}2020200\ldots \\ x^4 &= 0.222\textcircled{2}222002\ldots \end{aligned}$$

and define $x = .2020\ldots$ where the n^{th} entry of x, x_n, is the flipped entry of x_n^n, that is, if $x_n^n = 2$ then $x_n = 0$ and of $x_n^n = 0$ then $x_n = 2$. Then $x \neq x^n$ for any n due to this construction, but also $x \in F$, so we conclude it is impossible to enumerate all of F.

Another quick proof: for each number $x \in F$ with its base 3 representation, e.g., $x = 0.22020000220\ldots$, make the bijection mapping it to $x = 0.11010000110\ldots$ (map $0 \mapsto 0$ and $2 \mapsto 1$), which is now the binary expansion of any number between $[0, 1]$, hence we have the same cardinality as $[0, 1]$.

Problem 8: (10 pts). **Subject: Weak convergence, Riemann-Lebesgue, Hölder.** Let $0 \neq g \in L^p(\mathbb{R})$ for $1 < p < \infty$ be a fixed function, and $f_n(x) \stackrel{\text{def}}{=} g(x)\sin(n\pi x)$.

(1) (7 pts) Prove $f_n \rightharpoonup 0$.

Solution: Weak convergence in L^p means that for all $h \in L^q$ with $1/p + 1/q = 1$, we want $\int h f_n \to 0$. By Hölder, $hg \in L^1$, and then via Riemann-Lebesgue, we know its Fourier transform decays to 0, i.e., $\lim_{|\omega| \to \infty} \int h(x)g(x)e^{-i\omega x}\,dx = 0$. Using Euler's identity, this gives the desired result.

(2) (3 pts) Prove f_n does not converge strongly. (Remark: g is arbitrary — you cannot choose it. For example, g may be the indicator function of the real numbers on $[0, 1]$.)

Solution: Sketch (this was a hard problem, and no one got it right): C_c^{∞} is dense in L^p, so approximate g with $\widetilde{g}$ (and define $\widetilde{f_n} = \widetilde{g}\sin(n\pi x)$), where $\|g - \widetilde{g}\|_p < \epsilon/2$, and hence $\|f_n - \widetilde{f_n}\|_p < \epsilon/2$ as well. Then $\|f_n\|_p \ge \|\widetilde{f_n}\|_p - \|f_n - \widetilde{f_n}\|_p$ and as long as we chose ϵ appropriately (careful not to make it circular), we just need to show $\liminf_{n \to \infty} \|\widetilde{f_n}\|_p^p > \epsilon^p$.

Now that we have continuous functions, we can bound $\widetilde{g}$ from below with a characteristic function that is supported on some *interval* $[a, b]$ (we can ignore the rest of the function, since we are looking at its absolute value so the other parts cannot have a negative contribution), so essentially (let

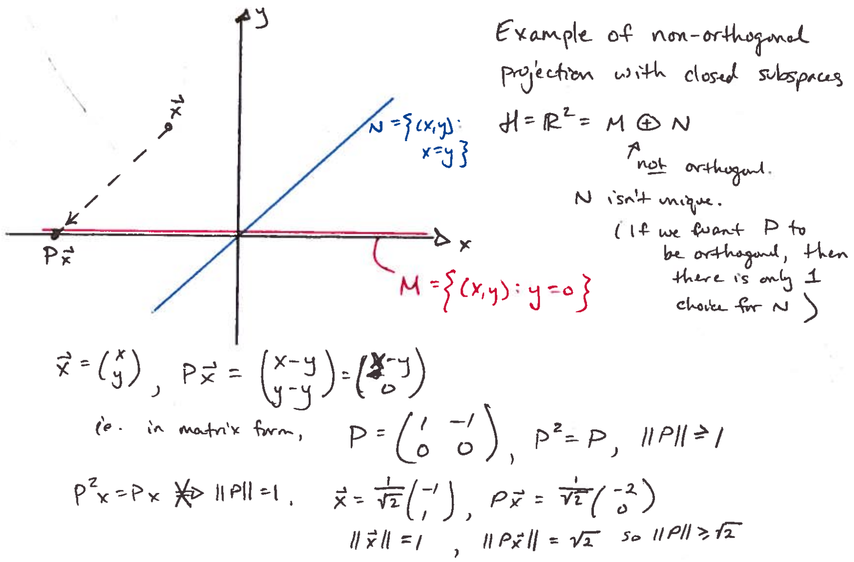

Figure 1: For the short-answer question about projections

$p = 2$ for simplicity) we need to show $c_n = \int_a^b \sin^2(nx)\,dx$ does not converge to 0 (we can ignore the π since a, b are arbitrary). We can actually evaluate this integral as $\frac{x}{2} - \frac{\sin(2nx)}{4n}\Big|_a^b$ so, $c_n = (b-a)/2 - \frac{1}{4n}(\sin(2nb) - \sin(2na))$, and since the sin terms are bounded, $c_n = (b-a)/2 + \mathcal{O}(1/n)$ so $c_n \to (b-a)/2 \neq 0$. The cases for other $1 < p < \infty$ are similar.

Problem 9: (1 pt) **Subject: Dual spaces, sequential continuity.** Prove that for $1 < p < \infty$, the dual of $\ell^p(\mathbb{N})$ is $\ell^q(\mathbb{N})$ where $p^{-1} + q^{-1} = 1$.

Solution: This was done on homeworks. Sketch: showing one is a subset of the other is quite easy and relies on Hölder's inequality; the other direction, you define a candidate vector in ℓ^q by looking at the action on the unit basis e_n, and prove it is equivalent using sequential continuity (e.g., that e_n is a Schauder basis for ℓ^p with $p < \infty$).