The fundamental identity for iterated spherical means and the inversion formula for diffraction tomography and inverse scattering

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A uniform derivation of the inversion formula for diffraction tomography and inverse scattering from the fundamental identity for iterated spherical means is presented.

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I. INTRODUCTION

In the book *Plane Waves and Spherical Means,*[1] John derives a relation between the iterated spherical mean and the spherical mean of a function. Using this relation, we have obtained a generalization of an important inversion formula that was recently derived by Devaney for use in diffraction tomography[2,3] and inverse scattering.[4]

Scattering experiments are usually designed to yield the scattered field as a function of two unit vectors which represent the direction of the incident wave and the direction at which the scattered field is recorded. The inversion formulae[2–4] reconstruct the scatterer from the measured scattered field which is a function of two unit vectors.

The relation we derive here can also be considered as the representation of a function in terms of the iterated spherical mean of its Fourier transform, provided the Fourier transform has compact support.

Our derivation is independent of the dimension of the space and thus generalizes the formulae derived in Refs. 2–4 to spaces of n dimensions.

II. THE FUNDAMENTAL IDENTITY FOR ITERATED SPHERICAL MEANS

Let f be a continuous function in a domain $D\subset R^n$. Set

$$I(x,r)=\frac{1}{\omega_n}\int_{|\xi|=1} f(x+r\xi)\,d\omega_\xi, \tag{1}$$

where $\omega_n=2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere in R^n, $x\in R^n$, $\xi\in R^n$ is a unit vector, and $d\omega_\xi$ is the solid angle differential form.

The function $I(x,r)$ is the average of the function f on a sphere of radius $|r|$ about the point x. The spherical mean $I(x,r)$ is even with respect to r.

Let us define the iterated spherical mean $M(x,\lambda,\mu)$ by the following formula:

$$M(x,\lambda,\mu)=\frac{1}{\omega_n}\int_{|\eta|=1} I(x+\lambda\eta,\mu)\,d\omega_\eta,$$

where λ,μ are real numbers and $\eta\in R^n$ is a unit vector. Alternatively, we can write

$$M(x,\lambda,\mu)=\frac{1}{\omega_n^2}\int_{|\xi|=1}\int_{|\eta|=1} f(x+\lambda\eta+\mu\xi)\,d\omega_\eta d\omega_\xi. \tag{2}$$

John obtained the fundamental identity for the iterated spherical mean $M(x,\lambda,\mu)$ (formula 4.9c[1]):

$$\begin{aligned}M(x,\lambda,\mu)&=[2\omega_{n-1}/(2\lambda\mu)^{n-2}\omega_n]\\&\quad\times\int_{\mu-\lambda}^{\mu+\lambda}[(r+\mu-\lambda)(r+\mu+\lambda)(\lambda+r-\mu)\\&\quad\times(\lambda-r+\mu)]^{(n-3)/2} rI(x,r)\,dr.\end{aligned} \tag{3}$$

In the next section we use this identity to derive a formula relating a function to the iterated spherical mean of its Fourier transform.

III. THE INVERSION FORMULA FOR DIFFRACTION TOMOGRAPHY AND INVERSE SCATTERING

We set $\lambda=\mu=k$ in (3) and obtain

$$\begin{aligned}M(x,k,k)&=\frac{\omega_{n-1}}{2^{n-3}k^{2n-4}\omega_n}\\&\quad\times\int_0^{2k}(4k^2-r^2)^{(n-3)/2}r^{n-2}(x,r)\,dr.\end{aligned} \tag{4}$$

Let $\hat F$ be a function with support inside the n-dimensional ball B_{2k},

$$B_{2k}=\{p:|p|<2k\}.$$

In scattering theory the surface of this ball is called the Ewald limiting sphere.

Consider the function

$$f_y(p)=[|p|/(4k^2-|p|^2)^{(n-3)/2}]\hat F(p)e^{ip\cdot y},$$

where $p\in R^n$ and $y\in R^n$ is a parameter.

We compute the spherical mean $I_y(0,r)$ for the function $f_y(p)$. From the definition (1) it follows that

$$I_y(0,r)=\frac{1}{\omega_n}\frac{r}{(4k^2-r^2)^{(n-3)/2}}\int_{|\xi|=1}\hat F(r\xi)e^{ir\xi\cdot y}\,d\omega_\xi.$$

Making use of relation (4), we find

$$\begin{aligned}M_y(0,k,k)&=\frac{\omega_{n-1}}{2^{n-3}k^{2n-4}\omega_n^2}\int_0^{2k}r^{n-1}\,dr\\&\quad\times\int_{|\xi|=1}\hat F(r\xi)e^{ir\xi\cdot y}\,d\omega_\xi.\end{aligned} \tag{5}$$

Since the support of the function $\hat F(p)$ is contained in the ball B_{2k} the function $F(y)$ defined by

$$F(y)=\frac{1}{(2\pi)^n}\int_{|p|<2k}\hat F(p)e^{ip\cdot y}\,dp \tag{6}$$

coincides with the inverse Fourier transform of $\hat F$. Thus we have from (5)

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$$M_y(0,k,k) = (8\pi^n \omega_{n-1}/k^{2n-4}\omega_n^2)F(y). \tag{7}$$

Also, from the definition (2) of the iterated spherical mean for the function $f_y(p)$ we obtain

$$M_y(0,k,k) = \frac{1}{k^{n-4}\omega_n^2}\int_{|\xi|=1}\int_{|\eta|=1}\frac{|\xi+\eta|}{(4-|\xi+\eta|^2)^{(n-3)/2}} \times \widehat{F}(k\xi + k\eta)e^{ik(\xi+\eta)\cdot y}\,d\omega_\eta d\omega_\xi. \tag{8}$$

We can replace the unit vector η by $-\eta$ in the formula (8). Comparing (7) and (8), we finally obtain the representation

$$F(y) = \frac{k^n}{8\pi^n\omega_{n-1}}\int_{|\xi|=1}\int_{|\eta|=1}\frac{|\xi-\eta|}{(4-|\xi-\eta|^2)^{(n-3)/2}} \times \widehat{F}(k\xi - k\eta)e^{ik(\xi-\eta)\cdot y}\,d\omega_\eta d\omega_\xi. \tag{9}$$

For $n = 2$ and for $n = 3$ the identity (9) reduces to the formulae obtained by Devaney.[2,4]

If the support of $\widehat{F}$ in (6) is not restricted to the ball B_{2k}, then (6) defines the so-called low-pass-filtered version of the function whose Fourier transform is $\widehat{F}$, and (9) provides a representation for the low pass filtered version of that function.

In diffraction tomography and in inverse scattering the function $\widehat{F}(k\xi - k\eta)$ represents the measured data.[2-4]

In the case of diffraction tomography the function $F(y)$ defined by (6) is the low-pass-filtered version of the two-dimensional object profile.[2]

In the case of inverse scattering the function $F(y)$ defined by (6) is the low-pass-filtered version of the so called interatomic distance function.[4]

In both cases formula (9) provides the basis for reconstruction algorithms.[2-4]

We note that formula (9) can also be written in the form

$$F(y) = \frac{k^n}{2^{(n+2)/2}\pi^n\omega_{n-1}}\int_{|\xi|=1}\int_{|\eta|=1}\frac{(1-\xi\cdot\eta)^{1/2}}{(1+\xi\cdot\eta)^{(n-3)/2}} \times \widehat{F}(k\xi - k\eta)e^{ik(\xi-\eta)\cdot y}\,d\omega_\eta d\omega_\xi. \tag{10}$$

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