Reconstructing discontinuities in multidimensional inverse scattering problems: smooth errors vs small errors

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Constructing parametrices instead of Green's functions to solve integral equations produces a new class of algorithms for nondestructive evaluation. These algorithms allow reconstruction of discontinuities of parameters of the physical medium in 2- and 3-D problems with variable background and arbitrary source-receiver geometry. An example of such algorithms is presented.

Many practical problems of nondestructive evaluation can be solved provided we can accurately reconstruct discontinuities of the parameters of the physical medium. Seismic exploration, medical applications, and crack and void detection are examples. Our concern is a mathematical formulation of the linearized inverse scattering problem so that we can (1) obtain explicit algorithms and (2) prove that indeed the discontinuities are recovered. The approach we take has two main features. First, all approximations realizing physical assumptions about the wave propagation are made in the direct problem, so that we reduce the inverse problem to solving a linear integral equation of the first kind with an oscillatory kernel. Second, to solve this integral equation we construct a parametrix solution, since, in general, it is impossible to construct explicitly the Green's function of a differential or an integral equation. The parametrix is a solution which is defined exactly as the Green's function except that an arbitrary smooth function may have been added to the source term of the equation. Parametrices are being widely used in the theory of pseudodifferential operators (see Ref. 1, for example) and represent a standard approach to gain insight into the properties of the partial differential equations. In dealing with the problems of nondestructive evaluation, we make use of these solutions to obtain algorithms for reconstructing parameters which describe the medium.[2] The error created by these algorithms is smooth (and slowly varying) rather than small, and, therefore, only the discontinuities of parameters are guaranteed to be recovered. We note, however, that this error also appears small in many important cases, such as in the case of a constant background.

To illustrate this approach, we consider a medium where wave propagation is described by the Helmholtz equation (a fluid with constant density, for example). Suppose the index of refraction in some region X is of the form $n^2(x) = n_0^2(x) + f(x)$, where $n_0(x)$—the background index of refraction—is known. Then the problem is to characterize the function $f(x)$—object profile—using observations of the (singly) scattered field on the boundary ∂X of the region X. Let the region X be three dimensional. (However, the specific dimension of X is not essential in our approach and enters only as a parameter.)

We linearize this inverse problem using the distorted wave Born approximation and the ray approximation for the Green's functions. Such a linearization yields [see Ref. 2, Eq. (2.12)] the integral representation of the singly scattered field due to a point source

$$v^{sc}(k,\xi,\eta) = -k^2 \int_X A^{\mathrm{out}}(x,\xi) \exp[ik\phi^{\mathrm{out}}(x,\xi)]f(x) \times A^{\mathrm{in}}(x,\eta) \exp[ik\phi^{\mathrm{in}}(x,\eta)]dx, \tag{1}$$

as a function of the receiver position ξ, the source position η, and wave number k. $\phi^{\mathrm{in}}(x,\eta),\phi^{\mathrm{out}}(x,\xi)$ are phase functions which satisfy the eikonal equations

$$[\nabla_x\phi^{\mathrm{in}}(x,\eta)]^2 = n_0^2(x),$$

$$[\nabla_x\phi^{\mathrm{out}}(x,\xi)]^2 = n_0^2(x).$$

For fixed x,ξ,η the phase functions $t^{\mathrm{in}} = \phi^{\mathrm{in}}(x,\eta)$ and $t^{\mathrm{out}} = \phi^{\mathrm{out}}(x,\xi)$ are travel times from the source location η to the point x inside the region X and from the point x to the receiver location ξ, respectively.

Functions A^{in}, A^{out} in Eq. (1) are amplitudes and satisfy the transport equations

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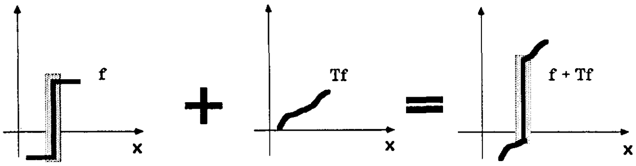

Fig. 1. Adding a smooth function changes neither the location nor the size of a jump discontinuity.

$$A^{\text{in}}(x,\eta)\nabla_x^2\phi^{\text{in}}(x,\eta) + 2\nabla_x A^{\text{in}}(x,\eta)\cdot\nabla_x\phi^{\text{in}}(x,\eta) = 0,$$

$$A^{\text{out}}(x,\xi)\nabla_x^2\phi^{\text{out}}(x,\xi) + 2\nabla_x A^{\text{out}}(x,\xi)\cdot\nabla_x\phi^{\text{out}}(x,\xi) = 0.$$

These equations describe how the amplitude of a signal changes along the rays connecting the source location η on the boundary ∂X with the point x inside the region X and the point x with the receiver location ξ on the boundary ∂X, respectively. Transport equations reduce to ordinary differential equations along rays,[3] and initial conditions should be added to compute amplitudes A^{in}, A^{out} in Eq. (1). If the background index of refraction $n_0(x)$ is discontinuous, the rays satisfy Snell's law on surfaces of discontinuities, and appropriate transmission coefficients have to be used in computing amplitudes on these surfaces.

The integral representation (1) is an integral equation for the unknown function f. The scattered field $v^{\text{sc}}(k,\xi,\eta)$ in Eq. (1) is a function of the wave number k and hence is related to the scattered field in the time domain via the Fourier transform. We assume that in experiments the scattered field is measured in the time domain, so that the function

$$u^{\text{sc}}(t,\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} v^{sc}(k,\xi,\eta)\exp(-ikt)dk$$

is given.

Our goal now is to find an approximation to the object profile f given the background index of refraction $n_0(x)$ and the singly scattered field $u^{\text{sc}}(t,\xi,\eta)$ in the time domain so that the error of the approximation is smooth. Let f_{mig} be a function which is constructed by some (as yet unspecified) method and suppose that

$$f_{mig} = f + Tf \tag{2}$$

for some smoothing operator T. The mathematical definition of classes of smoothing operators can be found in Refs. 1, 2, 4, and 5. Here we say that the operator T is a smoothing operator if the function Tf has at least one more continuous derivative than does the function f. Thereby f_{mig} contains all the information about the discontinuities of the function f, since $f_{mig} - f = Tf$ is smoother than f. This is illustrated in Fig. 1 for the 1-D case.

Let us describe the actual approximate solution to Eq. (1) for the case when the source position η is fixed. As shown in Ref. 2, a specific function f_{mig} can be explicitly constructed using the generalized backprojection (the dual of the generalized Radon transform[4,5]) of the singly scattered field

$$f_{mig}(x) = -\frac{1}{8\pi^2}\int_{\partial X} u^{\text{sc}}(t,\xi,\eta)\big|_{t=\phi^{\text{in}}(x,\eta)+\phi^{\text{out}}(x,\xi)} b(x,\xi)d\xi. \tag{3}$$

For a given point of reconstruction x and fixed source position η, we integrate the scattered field along the time–distance surface (curve in the 2-D case) $t = \phi^{\text{in}}(x,\eta) + \phi^{\text{out}}(x,\xi)$, which is dictated by the background index of refraction $n_0(x)$. It is clear that if there were a reflector at the point x, along this curve the scattered field is most affected. The weight function $b(x,\xi)$ in Eq. (3) is chosen so that we recover the jump of the function f at the point x as a result of such integration. The weight function $b(x,\xi)$ depends on x—the point of reconstruction—and is given by

$$b(x,\xi) = \frac{h(x,\xi)}{A^{\text{out}}(x,\xi)A^{\text{in}}(x,\eta)},$$

with

$$h(x,\xi)d\xi = n_0^3(1 + \cos\psi)d\omega, \tag{4}$$

where

$$\cos\psi(x,\xi,\eta) = \frac{\nabla_x\phi^{\text{out}}(x,\xi)\cdot\nabla_x\phi^{\text{in}}(x,\eta)}{n_0^2(x)}.$$

Here $\psi(x,\xi,\eta)$ is the angle between the two rays traced from the source and from the receiver to the point x, and $d\omega$ is the standard measure on the unit sphere. Equation (4) describes the rate of change at the point x of the direction of the ray connecting point x with the receiver with respect to the receiver position on the boundary ∂X. Evaluation of Eq. (3) for a general background index of refraction $n_0(x)$ can be achieved by ray tracing. In the case of a constant background the ray paths are straight lines, and, therefore, phase functions and amplitudes can be determined analytically (see Ref. 2 for a few examples).

It was proved in Ref. 2 that Eq. (3) is a parametrix of the integral equation (1), and the function f_{mig} satisfies Eq. (2), where T is a smoothing operator. Hence Eq. (3) can be used to reconstruct the discontinuities of the function f.

Algorithms analogous to Eq. (3) are derived when receiver positions are dependent on source positions.[2] The most simple case is the one where receivers and sources are coincident. Figures 2 and 3 adopted from Ref. 6 describe the results of computer simulations in this case. Figure 2 shows the 2-D configuration of an experiment and the generated singly scattered field. Coincident source–receiver positions are located along lines A, B, and C, which together form the boundary ∂X. The background index of refraction is a constant for simplicity of computations. The object consists of 18-point scatterers of equal reflectivity separated by approximately one wavelength (at the central frequency of the source) and distributed to form the letter S. The result of the reconstruction is shown in Fig. 3. In this synthetic example locations of the point scatterers and the jump of the index of refraction at these points are fully recovered.

In conclusion, we note that algorithms obtained by solving the linearized inverse problem in this manner are related to some of the *migration* schemes used in seismic exploration,[6] and the use of parametrix solutions allows extension of previous methods of recover-

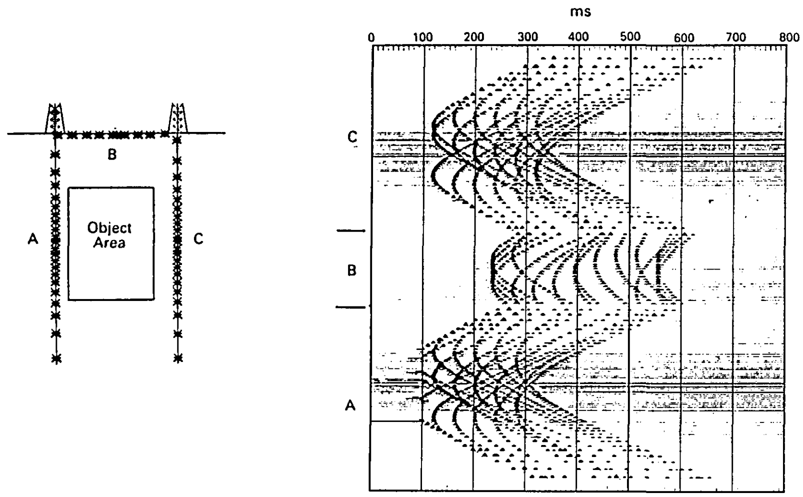

Fig. 2. Coincident source–receiver configuration and computer-simulated wave field generated by 18-point scatterers of equal reflectivity placed in the medium with constant index of refraction. (The scatterers form the shape of the letter S.)

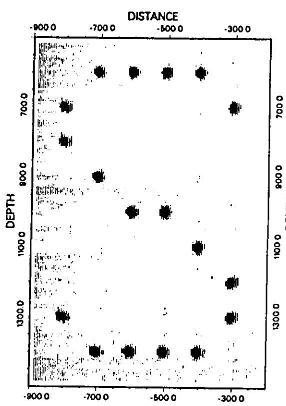

Fig. 3. Result of the reconstruction using the source–receiver configuration and the data shown in Fig. 2.

ing discontinuities such as migrations to the case of variable background and arbitrary configurations of sources and receivers.

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References

1. M. E. Taylor, *Pseudodifferential Operators* (Princeton U. P., Princeton, N.J., 1981).
2. G. Beylkin, "Imaging of Discontinuities in the Inverse Scattering Problem by Inversion of a Causal Generalized Radon Transform," J. Math. Phys. **26**, 99 (1985).
3. J. B. Keller, "Rays, Waves and Asymptotics," Bull. Am. Math. **84**, 727 (1978).
4. G. Beylkin, "Generalized Radon Transform and its Applications," Ph.D. Thesis, New York U. (1982).
5. G. Beylkin, "The Inversion Problem and Applications of the Generalized Radon Transform," Commun. Pure Appl. Math. **37**, 579 (1984).
6. D. Miller, M. Oristaglio, and G. Beylkin, "A New Formalism and Old Heuristic for Seismic Migration," in *Extended Abstract, Fifty-fourth Annual International Meeting of Society of Exploration Geophysics*, Atlanta (1984).