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Compactly Supported Wavelets Based on Almost Interpolating and Nearly Linear Phase Filters (Coiflets)

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New compactly supported wavelets for which both the scaling and wavelet functions have a high number of vanishing moments are presented. Such wavelets are a generalization of the so-called coiflets and they are useful in applications where interpolation and linear phase are of importance. The new approach is to parameterize coiflets by the first moment of the scaling function. By allowing noninteger values for this parameter, the interpolation and linear phase properties of coiflets are optimized. Besides giving a new definition for coiflets, a new system for the filter coefficients is introduced. This system has a minimal set of defining equations and can be solved with algebraic or numerical methods. Examples are given of the various types of coiflets that can be obtained from such systems. The corresponding filter coefficients are listed and their properties are illustrated.

1. INTRODUCTION

Among compactly supported wavelets for $L^2(\mathbf{R})$ a family known as *coiflets* has a number of properties that make it particularly useful in numerical analysis and signal processing [1, 8, 9]. Coiflets allow for both the scaling and the wavelet functions to have a high number of vanishing moments and, as we show here, the associated low-pass filters are almost interpolating and nearly linear phase within the passband. In 1989, R. Coifman

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suggested the design of orthonormal wavelet systems with vanishing moments for both the scaling and the wavelet functions. They were first constructed by Daubechies [9] and she named them *coiflets*.

In [1] shifted vanishing moments for the scaling function φ were used to obtain *one point quadratures*

$$f(x) \approx \sum_{k \in \mathbf{Z}} f(x_k)\varphi(x-k), \tag{1.1}$$

where f is a sufficiently smooth function on the multiresolution space V_0 and $\{f(x_k)\}$ are good approximations of the coefficients of f in the expansion.

Since in [1] both matrices and operators were considered, the points $\{x_k\}$ were chosen to be $x_k = k + \alpha$, where α is an integer. This "shift" α corresponds to the first moment of the scaling function φ,

$$\alpha = \int_{\mathbf{R}} x\varphi(x)\,dx. \tag{1.2}$$

Note that α is not the center of mass because $\varphi(x)$ is not a positive function [12], except for the Haar case.

The coiflets constructed by Daubechies correspond to particular integer choices of the shift α. Several other examples of coiflets, still for integer shifts, can be found in the literature [4, 11]. In this paper we use the fact that α does not have to be an integer. As a matter of fact, α may be chosen to be noninteger to optimize the construction of coiflets. An example of an approach similar to ours can be found in [15]. Furthermore, we show that the shift α cannot take arbitrary real values. In fact, we show that its value lies within the support of the scaling function. Therefore, the shifts for the known coiflets necessarily correspond to some integer values within this support.

Relation (1.1) is useful in pseudo-wavelet approaches to adaptively solving PDEs. Without going into details here (see [2]), let us state that if *both* f and f^2 belong to V_0, then

$$f^2(x) \approx \sum_{k \in \mathbf{Z}} f^2(x_k)\varphi(x-k) \tag{1.3}$$

is a quantifiable approximation. Notice that approximations like (1.1) or (1.3) are not valid for Fourier or Fourier-like bases.

On the other hand, equality in (1.1) cannot be achieved for all functions in V_0 by using any compactly supported wavelets. However, using infinite impulse response (IIR) filters, it is possible to have an exact version of (1.1) or (1.3) and this choice corresponds to *interpolating* filters.

A similar situation occurs if we require linear phase response, which is another desired property for the associated quadrature mirror filter (QMF) [3] of the wavelet bases. Except for the Haar system, finite impulse response (FIR) QMFs cannot have a linear phase response. To obtain that property one has to use IIR filters or give up orthogonality and replace it by biorthogonality.

In this paper we show that FIR *coiflets* can nearly achieve both properties, interpolation and linear phase, while keeping a reasonable number of vanishing moments for the

[3] Originally, these kind of filters leading to perfect reconstruction were named conjugate quadrature filters [16] while the denomination QMF from [10] would only apply to some aliasing cancelling filters. We use the term QMF as in [8, pp. 162, 163], where one can find a history of both terms.

wavelet ψ. The key to our approach is to insist on a reasonable approximation to linear phase only in the passband of the associated low-pass filter m_0.

It is well known that the properties defining coiflets can be easily described in terms of the coefficients $\{h_k\}$ of m_0. The conditions on $\{h_k\}$ turn out to be dependent [14], and one of the goals of this article is to derive a system that is free of redundant equations. To obtain such a system, we perform a change of variables on $\{h_k\}$ via a linear transformation that has the shift α as a parameter. This defining system is partly linear and partly quadratic. For filter lengths up to 20 the system can be explicitly solved via algebraic methods like Gröbner bases. Its particularly simple structure allows one to find all possible solutions. For longer filters we apply Newton's method to numerically compute some solutions. Nevertheless, for arbitrary filter lengths, we were unable to solve the open problem of the consistency of the defining system, i.e., we could not yet prove the existence of coiflets for an arbitrary number of vanishing moments.

We modify the original definition of coiflets in [1, 9] to allow for noninteger shifts α in (1.2) and to make more specific the relationship between the length of the low-pass filter and the number of vanishing moments of both the wavelet and scaling functions.

This paper is organized as follows. In Section 2 we give some preliminaries about wavelets in general. The moment conditions for both the scaling and wavelet functions are discussed in Section 3. In Section 4 we give a new definition of coiflets and motivate it. In Sections 5 and 6 we address two properties of coiflets: the interpolation property and nearly linear phase. We introduce the polyphase equation in Section 7 and use it in the construction of coiflets in the next section. Furthermore, in Section 8 we give details about the linear and quadratic equations of the defining system for coiflets. We also discuss the various types of coiflets that can be obtained from such systems and show explicit examples in Section 9. For clarity, we gathered auxiliary material in the Appendix.

2. PRELIMINARIES

- Unless otherwise indicated, x and ξ are real variables while z is a complex variable.
- A QMF is a 2π-periodic function m_0,

$$m_0(\xi) = \sum_{k\in\mathbf{Z}} h_k e^{-ik\xi}, \tag{2.1}$$

such that

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1. \tag{2.2}$$

The numbers $\{h_k\}$ are the *coefficients* of the filter m_0. We assume that all h_k are real and only a finite number of them is nonzero. The QMF condition (2.2) is then equivalent to

$$2\sum_k h_k h_{k+2n} = \delta_{no} \qquad \text{for } n \in \mathbf{Z}. \tag{2.3}$$

The Kronecker symbol δ_{nm} is defined as $\delta_{nm} = 1$ if $m = n$ and $\delta_{nm} = 0$ otherwise.

We denote by H the symbol of $\{h_k\}$, i.e., the *transfer function* of m_0. We have $H(e^{-i\xi}) = m_0(\xi)$ or

$$H(z) = \sum_k h_k z^k. \tag{2.4}$$

We also refer to such H as a QMF. As a consequence of (2.3), H satisfies the following functional equation:

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1. \tag{2.5}$$

We refer to this equation as the QMF equation.

In order to generate a regular multiresolution analysis (see [5, 6]), we need two additional properties for H.

The first one is the *normalization* or *low-pass* condition. It forces

$$m_0(0) = 1 \qquad \text{or} \qquad H(1) = 1. \tag{2.6}$$

The second one, which we refer to as Cohen's condition, ensures that H is nonzero in certain locations on the unit circle [7].

In practice, we first find a normalized H satisfying the QMF equation and then verify Cohen's condition.

- A solution φ of

$$\varphi\left(\frac{x}{2}\right) = 2\sum_k h_k \varphi(x-k) \tag{2.7}$$

is called a "scaling function." Equivalently, on the Fourier side we have

$$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi), \tag{2.8}$$

where $\hat{\varphi}(\xi) = \int_{-\infty}^{+\infty} \varphi(x) \mathrm{e}^{-i\xi x}\, dx$, and

$$\hat{\varphi}(0) = 1, \tag{2.9}$$

as a consequence of (2.6).

3. MOMENT CONDITIONS

One of the key properties of interest for wavelet bases [1, 13] is the property of vanishing moments of the wavelet ψ:

$$\int_{\mathbf{R}} x^k \psi(x)\, dx = 0 \qquad \text{for } 0 \le k < M. \tag{3.1}$$

In [8, Theorem 5.5.1], it is shown that if the wavelet has m bounded derivatives then we have at least m vanishing moments, i.e., we have $M > m$ in (3.1).

Also, (3.1) implies that all polynomials p of degree less than M can be expressed as linear combinations of integer translates of φ,

$$p(x) = \sum_k \left(\int_{\mathbf{R}} p(y)\varphi(y-k)\, dy \right) \varphi(x-k). \tag{3.2}$$

See [13, Section 2.6] or [17] for details.

In terms of the symbol H, (3.1) requires that

$$\sum_j (-1)^j j^k h_j = 0 \qquad \text{for } 0 \le k < M, \tag{3.3}$$

or equivalently, the factorization

$$H(z) = \left(\frac{1+z}{2}\right)^M Q(z), \tag{3.4}$$

where $Q(-1) \neq 0$.

As pointed out in the Introduction, we are interested in vanishing (shifted) moments of the scaling function

$$\mathcal{M}^{\varphi}_{\alpha,k} = \int_{\mathbf{R}} (x-\alpha)^k \varphi(x)\, dx = \delta_{k0} \qquad \text{for } 0 \le k < N, \tag{3.5}$$

where α is a real number.

If α is 0, we write $\mathcal{M}^{\varphi}_k$ for the kth moment of φ. We have

$$\mathcal{M}^{\varphi}_{\alpha,n} = \sum_{k=0}^{n} \binom{n}{k} (-\alpha)^{n-k} \mathcal{M}^{\varphi}_k. \tag{3.6}$$

From (2.8) it follows that

$$e^{2i\alpha\xi}\hat{\varphi}(2\xi) = e^{i\alpha\xi} H(e^{-i\xi}) e^{i\alpha\xi}\hat{\varphi}(\xi), \tag{3.7}$$

and then, by taking derivatives at $\xi = 0$,

$$(2^n - 1)\mathcal{M}^{\varphi}_{\alpha,n} = \sum_{k=0}^{n-1} \binom{n}{k} \mathcal{M}^{h}_{\alpha,n-k} \mathcal{M}^{\varphi}_{\alpha,k}. \tag{3.8}$$

In addition, from (2.9)

$$\mathcal{M}^{\varphi}_{\alpha,0} = 1. \tag{3.9}$$

Here $\mathcal{M}^{h}_{\alpha,k}$ are the shifted moments of the sequence $\{h_j\}$:

$$\mathcal{M}^{h}_{\alpha,k} = \sum_j (j-\alpha)^k h_j. \tag{3.10}$$

Again, for $\alpha = 0$ we drop the index α and denote the nth moment by $\mathcal{M}^{h}_n$.

Because of the recurrence (3.8), (3.9), the moments of φ can be computed using the moments of $\{h_k\}$. Nevertheless, if some of the moments of φ are zero, we also obtain the following explicit relation.

LEMMA 3.1. *Assume* $\mathcal{M}^{\varphi}_{\alpha,k} = \delta_{k0}$ *for all* $k,\ 0 \le k < N$. *Then*

$$\mathcal{M}^{\varphi}_{\alpha,n} = \frac{1}{2^n - 1}\mathcal{M}^{h}_{\alpha,n} \tag{3.11}$$

for $0 < n < 2N$.

Equations (3.6) and (3.8) imply that the following four conditions, valid for all $k,\ 0 \leq k < N$, are equivalent:

$$\mathcal{M}_k^{\varphi} = \int_{\mathbf{R}} x^k \varphi(x)\, dx = \alpha^k, \tag{3.12}$$

$$\mathcal{M}_{\alpha,k}^{\varphi} = \int_{\mathbf{R}} (x-\alpha)^k \varphi(x)\, dx = \delta_{k0}, \tag{3.13}$$

$$\mathcal{M}_{\alpha,k}^{h} = \sum_j (j-\alpha)^k h_j = \delta_{k0}, \tag{3.14}$$

$$\mathcal{M}_k^{h} = \sum_j j^k h_j = \alpha^k. \tag{3.15}$$

Therefore, imposing moment conditions for either the wavelet or the scaling function amounts to finding a QMF H with moment conditions for its sequence of coefficients. In particular, the first moment of φ, as defined in (1.2), equals the derivative of H at one,

$$\alpha = H'(1).$$

On the other hand, (2.1) forces $|H(z)| \leq 1$ for z on the unit circle. These last two properties allow us to show that the value α should be within the support of φ. Observe that this result is not evident since φ is not a positive function.

PROPOSITION 3.2. *Let $H(z) = \sum_{k=0}^{n} h_k z^k$ be any nonconstant polynomial with real coefficients and $h_0 h_n \neq 0$.*

$$\textit{If} \qquad H(1) = 1 \qquad \textit{and} \qquad \sup_{|z|=1} |H(z)| \leq 1,$$

then $H'(1)$ belongs to the interval $(0, n)$.

Proof. We need the following version of the classical Bernstein inequality for trigonometric polynomials (see [21, Theorem 7.24] or [3, Corollary 5.1.6]): *Let p be any polynomial with complex coefficients and at most degree n. Then* $\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|$. *Equality holds iff there exists a constant c such that $p(z) = cz^n$.*

We apply Bernstein's inequality to the polynomials $H(z)$ and $z^n H(z^{-1})$. In both cases equality cannot hold and we obtain $|H'(1)| < n$ and $|n - H'(1)| < n$, respectively. Since H has real coefficients, $H'(1)$ is a real number and the proposition follows. ∎

4. COIFLETS

As argued above, it suffices to define coiflets in terms of the filter H. Since for any integer n the filter $z^n H(z)$ generates the same multiresolution analysis as H, we always assume the coefficients $\{h_k\}$ of H to be zero for $k < 0$.

DEFINITION 4.1 (coiflets). Let $\{h_j\}_{j=0}^{L-1}$ be the coefficients of a real QMF H. We say that H is a *coiflet* of shift α and moments M, N if the following three conditions are satisfied:

$$\sum_{j=0}^{L-1}(-1)^j j^k h_j = 0 \qquad \text{for } 0 \le k < M, \tag{4.1}$$

$$\sum_{j=0}^{L-1} j^k h_j = \alpha^k \qquad \text{for } 0 \le k < N, \tag{4.2}$$

$$3M > L - 1 \qquad \text{and } 3N \ge L - 1. \tag{4.3}$$

Using the equations of the previous section, if a *coiflet* H also satisfies Cohen's condition, its associated wavelet and scaling functions will have M, respectively $N - 1$, vanishing moments. The normalization $H(1) = 1$ corresponds to (4.2) with $k = 0$.

The case $L = 2$ corresponds to the Haar basis. For $L = 4$, (4.3) forces $M > 1$ and therefore coiflets of length four are the same as Daubechies' maximally flat filters of that length. In Section 9 we discuss the cases $L = 8$ and $L = 18$.

Our definition of coiflets is restrictive in that we require not just some but nearly all possible vanishing moments for both the scaling and the wavelet functions (see Remarks 4.3 below).

It follows from Proposition 3.2 that the value of the shift α belongs to $(0, L - 1)$. Furthermore, in all cases computed, there were regions in $(0, L - 1)$ where α did not occur. We refer to Table 1 to illustrate this fact for integer shifts. For example for $L = 14$, the values $\alpha = 1, 2$, and 6 are missing in the interval $(0, \frac{1}{2}(L - 1)) = (0, 6.5)$. Due to symmetry about the center 6.5 (see Section 8.3.1), the values $\alpha = 7$, 11, and 12 do not occur either.

It is important to realize that the conditions (4.1) and (4.2) are dependent. In fact, using the notation $[a]$ for the integer part of a, we have the following lemma from [14].

LEMMA 4.2. *Let H be a QMF with coefficients $\{h_j\}$ that satisfy $\sum_j j^k h_j = \alpha^k$ for $0 \le k < N$, then*

$$\sum_j (-1)^j j^k h_j = 0 \qquad \textit{for } 0 \le k < \left[\frac{1}{2}(N+1)\right].$$

TABLE 1
Coiflets with Integer Shifts

Length L	Shifts in $(0, \frac{1}{2}(L-1))$	M	N
6	{1, 2}	2	3
8	{1, 2, 3}	3	3
10	{1, 2, 3, 4}	4	3
12	{3, 4, 5}	4	5
14	{3, 4, 5}	5	5
16	{3, 4, 5, 6, 7}	6	5
18	{5, 6, 7}	6	7
20	{5, 6, 7, 8}	7	7
22	{5, 6, 7, 8, 9, 10}	8	7
24	{6, 7, 8, 9, 10}	8	9
26	{7, 8, 9, 10}	9	9
28	{8, 9, 10, 11, 12}	10	9

Proof. Applying the operator $(xD)^n$ (defined in the Appendix) at $z = 1$ to the QMF equation (2.5), or taking derivatives at $\xi = 0$ in (2.2), we have for all n,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_{n-k} a_k + \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_{n-k} b_k = \delta_{n0}, \tag{4.4}$$

where $a_k = (xD)^k H(1)$ and $b_k = (xD)^k H(-1)$.

The assumption on H implies that $b_0 = 0$ and that the first sum in (4.4) is zero for $n < N$. Choosing n even and $0 \leq n < N$ in the second sum, it follows that $b_k = 0$ for $0 \leq k < [\frac{1}{2}(N+1)]$. ■

Remarks 4.3. 1. The lemma shows why the condition (4.3) is consistent with the theory of polynomial QMF. For a QMF of degree $L - 1$ it is well known that there are only $\frac{L}{2}$ degrees of freedom for the filter coefficients (see [19] for example).

By asking $N \approx \frac{L}{3}$ we already have $M > \frac{L}{6}$. The $\frac{L}{6}$ extra conditions in (4.3) bring the total number of conditions to $\frac{L}{3} + \frac{L}{6} = \frac{L}{2}$. Viewed this way, *coiflets* are meant to maximize both numbers of vanishing moments, while their values remain close to each other.

2. If N is even and $M > \frac{N}{2}$ in the definition of coiflets, we obtain $\sum_j j^N h_j = \alpha^N$ (replace n by N in (4.4)). For this reason, N is always odd in our examples (see Tables 1, 2, and 6).

In particular, if $M > 1$ in (4.1) and $\int_{\mathbf{R}} x\varphi(x)\,dx = \alpha$ then

$$\int_{\mathbf{R}} x^2 \varphi(x)\,dx = \alpha^2.$$

This result has been noted by other authors. See, for example, [18, Theorem 2.3] or [11, Theorem 1].

3. We can give a geometric interpretation of Lemma 4.2. Condition (4.2) forces $|m_0|^2$ to be flat at zero. Because of the QMF condition, the same is true at π, and therefore m_0 is also flat at π, but only “half as flat."

5. ALMOST INTERPOLATING PROPERTY

Consider the scaling function φ associated with a *coiflet* of shift α and moments M, N.

Recall that each multiresolution space $\mathbf{V}_n$ is generated by the basis functions $\{\varphi_{nk}(x) = 2^{n/2}\varphi(2^n x - k)\}_k$. With (4.2), or equivalently (3.12), for any polynomial p of degree less than N,

$$\int p(x)\varphi_{nk}(x)\,dx = 2^{-n/2} \int p\left(\frac{y+k}{2^n}\right)\varphi(y)\,dy = 2^{-n/2} p\left(\frac{\alpha+k}{2^n}\right).$$

If the degree of p is also less than M, then (3.2) implies that

$$p(x) = \sum_k \left(\int_{\mathbf{R}} p(y)\varphi_{nk}(y)\,dy \right) \varphi_{nk}(x). \tag{5.1}$$

Now, assume that $p(x)$ is a polynomial of degree less than M and N. Combining both equations above, the coefficients in the expansion of such a polynomial (at any scale) are its values on a shifted dyadic grid:

$$p(x) = \sum_k p\left(\frac{\alpha + k}{2^n}\right)\varphi(2^n x - k).$$

Since at some scale any smooth function can be well approximated by polynomials, we have the almost interpolating property discussed in the Introduction.

Here we see the advantage of having both M and N as large as possible for a given filter length, but also of having their values close to each other.

6. NEARLY LINEAR PHASE PROPERTY

In this section, for a filter $H(e^{i\omega})$, we relate the condition of having vanishing moments with its phase being close to linear in the passband.

LEMMA 6.1. *Let $f(\xi)$ be a function that takes complex values and such that $f(-\xi) = \overline{f(\xi)}$. Assume that $f(0) = 1$ and consider the polar decomposition of f,*

$$f(\xi) = a(\xi)e^{ip(\xi)}, \tag{6.1}$$

in a neighborhood of $\xi = 0$. Because of the condition on f, a is an even and p an odd function. If for $\gamma \in \mathbf{R}$ and for all n, $0 < n < N$,

$$D^n(e^{-i\gamma\xi} f(\xi))(0) = 0, \tag{6.2}$$

then for $0 \le n < N$, the derivatives of p at 0 can be computed as

$$D^{2n+1}p(0) = \gamma\delta_{n0} - i\,D^{2n+1}(e^{-i\gamma\xi} f(\xi))(0). \tag{6.3}$$

Consequently, $p(\omega) = \gamma\omega + o(\omega^{2[N/2]})$ as $\omega \to 0$.

Proof. From (6.1)

$$\ln(e^{-i\gamma\xi} f(\xi)) = \ln(a(\xi)) + (p(\xi) - \gamma\xi)i. \tag{6.4}$$

Note that if a function g satisfies $D^k g(a) = \delta_{k0}$ for $0 \le k < N$, then the derivatives of the composition $h \circ g$ are given by

$$D^n(h \circ g)(a) = Dh(g(a))\, D^n g(a) \qquad \text{for } 0 < n < 2N. \tag{6.5}$$

Thus for $0 < n < 2N$, the nth derivative of the left-hand side in (6.4) equals $D^n(e^{-i\gamma\xi} f(\xi))$. The result then follows because $\ln a(\xi)$ is an even function. ■

Since both $H(e^{i\xi})$ and $\hat{\varphi}(\xi)$ (use (3.13) and (3.14)) satisfy the conditions of the previous lemma for $\gamma = \alpha$ or $-\alpha$, using Lemma 3.1 we arrive at the following proposition.

PROPOSITION 6.2. *Let H be a polynomial with real coefficients $\{h_k\}$ and moment conditions $\mathcal{M}^h_{\alpha,k} = \delta_{k0}$ for $0 \le k < N$. If φ is the scaling function solution of* (2.7) *and in* *a neighborhood of* $\xi = 0$

$$H(e^{i\xi}) = a_H(\xi)e^{ip_H(\xi)} \qquad \textit{and} \qquad \hat{\varphi}(\xi) = a_{\hat{\varphi}}(\xi)e^{ip_{\hat{\varphi}}(\xi)},$$

where a_H *and* $a_{\hat{\varphi}}$ *are real even functions and* p_H *and* $p_{\hat{\varphi}}$ *are real odd functions, then for* $0 \leq n < N$,

$$D^{2n+1}p_H(0) = \alpha\delta_{n0} + (-1)^n \mathcal{M}^h_{\alpha,2n+1}, \qquad \textit{and}$$

$$D^{2n+1}p_{\hat{\varphi}}(0) = -\alpha\delta_{n0} + (-1)^n \frac{\mathcal{M}^h_{\alpha,2n+1}}{2^{2n+1}-1}.$$

Consequently, $p_H(\omega) = \alpha\omega + o(\omega^{2[N/2]})$, *and* $p_{\hat{\varphi}}(\omega) = -\alpha\omega + o(\omega^{2[N/2]})$ *as* $\omega \to 0$.

As stated in the Introduction, a high number N of shifted vanishing moments for the scaling function implies that the phase of the associated filter is close to linear within the passband. The larger the value of N the better the approximation. The same considerations hold for the phase of $\hat{\varphi}$ but they do not necessarily apply to the phase of $\hat{\psi}$.

Recall that

$$\hat{\psi}(2\xi) = m_1(\xi)\hat{\varphi}(\xi), \qquad \text{where} \qquad m_1(\xi) = -\mathrm{e}^{-\mathrm{i}\xi}\overline{m_0(\xi+\pi)}.$$

This dependence of $p_{\hat{\psi}}$ on p_H and $p_{\hat{\varphi}}$ can be seen by comparing the top parts of Figs. 1 and 2. In the latter case, $p_{\hat{\psi}}$ is flatter at zero because of the better behavior at π of the corresponding p_H.

In Figs. 5–6, 9–10, and 13–14 one can see the effect of N on the phase and group delay for several filters and wavelets. In these examples, the values of N are 3, 5, and 9. Note that, as N increases, the filter group delay becomes flatter about zero.

Figures 1 and 2 compare the group delay of some coiflets of length 18 with Daubechies' maximally flat filters of the same length.

Since for Daubechies' filters $N = 3$ independent of their length, those filters cannot have group delays that are flat at zero. Still, some choices are better than others and the

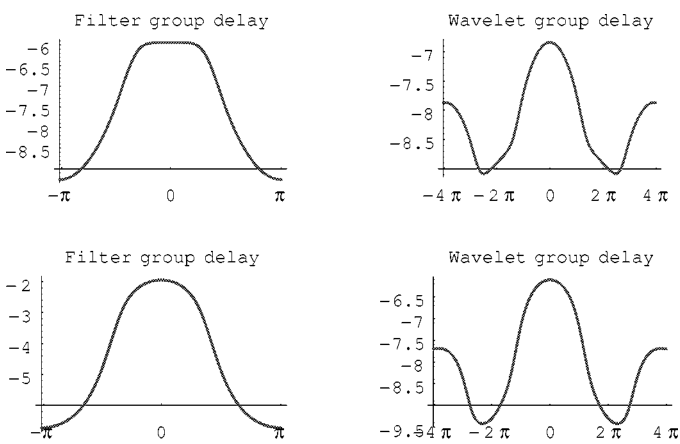

FIG. 1. Comparison between the group delays of m_0 and $\hat{\psi}$. Maximal coiflet for wavelet: length 18, case a (top) and Daubechies' extremal phase filter of the same length (bottom).

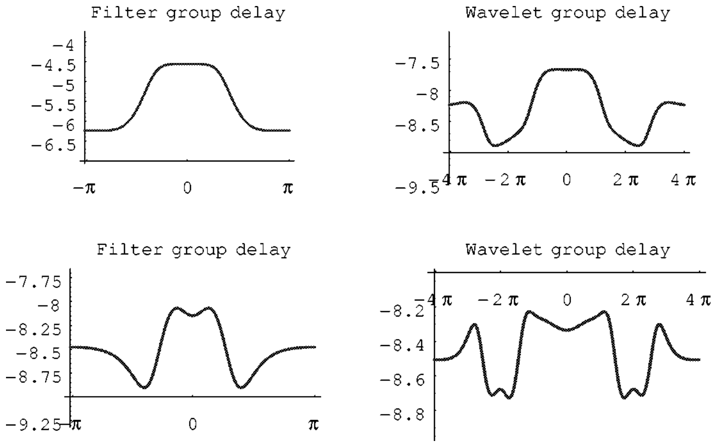

FIG. 2. Comparison between the group delays of m_0 and $\hat{\psi}$. Maximal coiflet for wavelet: length 18, case b (top) and Daubechies' least asymmetric filter of the same length (bottom).

least asymmetric filter in Fig. 2 is defined as the maximally flat filter whose phase is as linear as possible within the whole band $[-\pi, \pi]$. See [8, Section 8.1.1].

On the other hand, Proposition 6.2 implies that for coiflets only the phase in the passband is forced to be linear. Fortunately, the value of the phase in the stopband can be ignored in practice because the absolute value of the filter is close to zero in that region. The larger M is in (4.1), the more precise the last statement is, and we can again see the interplay of the conditions in the definition of coiflets.

7. THE POLYPHASE EQUATION

In order to find solutions of the QMF equation, we use an equivalent functional equation. Using the standard notation for the *polyphase components* H_0 and H_1 of $H(z)$,

$$H_0(z) = \sum_k h_{2k} z^k \qquad \text{and} \qquad H_1(z) = \sum_k h_{2k+1} z^k,$$

these functions satisfy

$$H_0(z^2) = \frac{H(z) + H(-z)}{2}, \tag{7.1}$$

$$H_1(z^2) = \frac{H(z) - H(-z)}{2z}, \tag{7.2}$$

$$H(z) = H_0(z^2) + z H_1(z^2). \tag{7.3}$$

Using the notation $\tilde{f}(z) = f(z^{-1})$ and (7.1), we have that (2.5) is equivalent to

$$(H\tilde{H})_0(z) = \frac{1}{2}, \tag{7.4}$$

and since

$$H\tilde{H} = (H_0\tilde{H}_0 + H_1\tilde{H}_1)(z^2) + z(\tilde{H}_0 H_1 + z^{-1} H_0 \tilde{H}_1)(z^2), \tag{7.5}$$

we obtain the *polyphase* equation

$$H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = \frac{1}{2}. \tag{7.6}$$

The problem of finding a solution H of the QMF equation (2.5) is thus replaced by finding the solutions H_0 and H_1 of the *polyphase* equation. Instead of performing two operations on the variable z in (2.5), namely $-z$ and z^{-1}, in (7.6) we only have z^{-1}.

8. THE CONSTRUCTION OF COIFLETS

Recall that we can write any polynomial QMF as $H(z) = \sum_{k=0}^{L-1} h_k z^k$, where $h_0 h_{L-1} \neq 0$.

We describe a system for coiflets not in terms of $\{h_k\}$ but in terms of the new variables

$$a_k = \frac{1}{k!}\sum_j \left(j - \frac{\alpha}{2}\right)^k h_{2j} \qquad \text{and} \qquad b_k = \frac{1}{k!}\sum_j \left(j - \frac{\alpha-1}{2}\right)^k h_{2j+1},$$

where $0 \le k \le l$, and $l = \frac{1}{2}(L-2)$. The transformation from $\{h_k\}$ to $\{a_k, b_k\}$ is linear and parameterized by α. As before, $\alpha = \sum_j j h_j$ is the first moment of φ.

For what follows, it is more convenient to describe a_k and b_k for arbitrary $k \ge 0$, using the operator xD. We then have

$$a_k = \frac{1}{k!}(xD)^k (x^{-\alpha/2} H_0)(1) \qquad \text{and} \qquad b_k = \frac{1}{k!}(xD)^k (x^{-(\alpha-1)/2} H_1)(1).$$

We denote by $\mathcal{V}$ the set of variables $\{a_k\}_{k=0}^{l}$ and $\{b_k\}_{k=0}^{l}$. In Lemma A.1 in the Appendix we show that H_0 and H_1, and therefore H, are completely determined by $\mathcal{V}$.

Note that a_k and b_k are not necessarily zero for $k > l$ but each of them can be expressed as a linear combination of the variables in $\mathcal{V}$. To verify that, apply part D of Lemma A.1 to the polynomials H_0 and H_1, which are both of degree l.

8.1. Quadratic Conditions

In order to impose the nonlinear (quadratic) conditions in (2.3), we use the equivalent formulation given by the polyphase equation.

Regarding (7.6),

$$S(z) = z^{(1/2)(L-2)}\left(H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) - \frac{1}{2}\right)$$

is a polynomial of degree at most $L-2$. Therefore, for the polyphase equation to hold, it suffices to show that

$$D^k S(1) = 0 \qquad \text{for } 0 \le k \le L-2.$$

Using (A.1) and (A.4) from the Appendix, the last equation is equivalent to

$$\frac{1}{2}\delta_{n0} = \frac{1}{n!}(xD)^n (H_0\tilde{H}_0 + H_1\tilde{H}_1)(1)$$

$$
\begin{aligned}
&= \frac{1}{n!}(xD)^n \left(x^{-\alpha/2}H_0(x)x^{\alpha/2}H_0(x^{-1}) + x^{-(\alpha-1/2)}H_1(x)x^{(\alpha-1)/2}H_1(x^{-1})\right)(1) \\
&= \sum_{k=0}^{n}(-1)^k(a_{n-k}a_k + b_{n-k}b_k) \qquad (8.1)
\end{aligned}
$$

for $0 \leq n \leq L-2$.

If n is odd, the previous equation is always satisfied and then, as we remarked earlier, $\frac{L}{2}$ equations are enough to characterize a QMF of length L.

8.2. Linear Conditions

We now discuss how to rewrite the (linear) conditions (4.1) and (4.2) for coiflets in terms of the variables in $\mathcal{V}$.

First, for $0 \leq k < M$, Eq. (4.1) is equivalent to

$$(xD)^k H(-1) = 0 \Leftrightarrow (xD)^k(x^{-\alpha}H(-x))(1) = 0$$

and, for $0 \leq k < N$, Eq. (4.2) is equivalent to

$$(xD)^k H(1) = \alpha^k \Leftrightarrow (xD)^k(x^{-\alpha}H(x))(1) = \delta_{k0}.$$

From (7.3), for x in a neighborhood of 1,

$$x^{-\alpha}H(-x) = \left(x^{-\alpha/2}H_0(x) - x^{-(\alpha-1)/2}H_1(x)\right)(x^2)$$

and

$$x^{-\alpha}H(x) = \left(x^{-\alpha/2}H_0(x) + x^{-(\alpha-1)/2}H_1(x)\right)(x^2).$$

Then,

$$\frac{1}{n!}(xD)^n(x^{-\alpha}H(-x))(1) = 2^n(a_n - b_n) \qquad (8.2)$$

and

$$\frac{1}{n!}(xD)^n(x^{-\alpha}H(x))(1) = 2^n(a_n + b_n). \qquad (8.3)$$

Therefore, the moment conditions for coiflets imply

$$\begin{cases} a_0 = b_0 = \frac{1}{2}, \\ a_1 = \cdots = a_{m-1} = b_1 = \cdots = b_{m-1} = 0, \end{cases} \qquad (8.4)$$

where $m = \min\{M, N\}$. For $k \geq m$, Eq. (4.1) implies that $a_k = b_k$, whereas (4.2) implies that $a_k = -b_k$.

Thus, we can rewrite the vanishing moment conditions on the sequence $\{h_k\}$ as very simple conditions on the variables in $\mathcal{V}$.

Substituting (8.4) into the system (8.1) we automatically verify the first m equations. That is, when $0 \leq n < m$ in (8.1) we obtain the dependence between linear and nonlinear equations described in Lemma 4.2.

8.3. A System for Coiflets

Combining (8.1) and (8.4), the system for coiflets can be written in terms of the unknowns $\{\alpha, a_m, \ldots, a_l, b_m, \ldots, b_l\}$,

$$\frac{1}{2}\delta_{n0} = a_n + b_n + \sum_{k=m}^{n-1}(-1)^k(a_{n-k}a_k + b_{n-k}b_k), \qquad n \text{ even}, \; m \le n \le L-2. \tag{8.5}$$

Again, $m = \min\{M, N\}$, where M, N, L satisfy (4.3), and $l = \frac{1}{2}(L-2)$. Recall that a_k and b_k for $k > l$ can be expressed in terms of the variables in $\mathcal{V} = \{a_0, \ldots, a_l, b_0, \ldots, b_l\}$.

Depending on whether m is even or odd, we are left with $\frac{1}{2}(2l + 1 - m - 1) = \frac{1}{2}(L-m-2)$ or $\frac{1}{2}(2l+1-m) = \frac{1}{2}(L-m-1)$ equations in (8.5). Remarkably, because of (8.4), the first half of these equations is linear. The other half is quadratic in the unknowns (as in the original QMF system). See the examples in Section 9.

Because of (4.3), we can check that the difference between the number of unknowns $\{\alpha, a_m, \ldots, a_l, b_m, \ldots, b_l\}$ and the number of equations in (8.5) is one.

Adhering to our definition, we are led to a one-parameter family of coiflets with parameter α. However, as can be seen in the examples in Section 9, the values of α are not completely arbitrary. As a matter of fact, they are restricted to certain regions.

8.3.1. Symmetry about $\frac{1}{2}(L-1)$. Let H be the QMF that defines a coiflet of length L, shift α, and moments M, N. From (2.5) it is clear that the reciprocal polynomial of H,

$$H_r(z) = z^{L-1}H(z^{-1}),$$

is also a QMF of length L. The coefficients of H_r are $\{h_{L-1}, \ldots, h_0\}$ and the associated scaling function is $\varphi_r(x) = \varphi(L-1-x)$. Note that H_r is also a coiflet with moments M, N but shift $L-1-\alpha$. Indeed, since H and H_r have the same multiplicity of zeros at -1, (4.1) follows. With respect to (4.2),

$$\begin{aligned}(xD)^n H_r(1) &= \sum_{k=0}^{n}\binom{n}{k}(L-1)^{n-k}(-1)^k(xD)^k H(1)\\ &= (L-1-\alpha)^n \qquad \text{if } 0 \le k < N.\end{aligned}$$

Finally, H_r cannot have more vanishing moments for φ_r because $H = (H_r)_r$, and a computation similar to the one above would force extra vanishing moments on φ.

Due to this symmetry, we can consider coiflets H whose shifts belong to the interval $(0, \frac{1}{2}(L-1)]$. All other coiflets correspond to the reciprocals H_r.

8.3.2. The non-maximal case. Given a QMF H of length L, we want to simultaneously satisfy (4.1)–(4.3) with the smallest possible M and N. A filter of that type will be called a *nonmaximal* coiflet or simply *coiflet*. This condition does not uniquely determine H, but as pointed out above, we have a one-parameter family of nonmaximal coiflets. We select the shift α as the parameter to characterize that family.

Within coiflets of a certain degree, we distinguish two cases: coiflets with integer shifts and maximal coiflets.

8.3.3. Coiflets with integer shifts. Coiflets for *integer* choices of the shift α were first computed by Daubechies [9]. In all cases that we computed, coiflets with integer shifts were always nonmaximal. In Table 1 we list, for different lengths L, the range of possible integer shifts in $(0, \frac{1}{2}(L-1))$ together with the corresponding number of vanishing moments: M for the wavelet function and N for the scaling function. Note that M and N remain the same for all the shifts, but the number of solutions may vary. For example, for length $L = 8$, there are three possible shifts, $\alpha = 1$, 2, and 3, and each has two possible solutions. For $L = 16$ the possible shifts are $\alpha = 3$, 4, 5, 6, and 7, with 2, 4, 2, 6, and 4 solutions, respectively. In other words, even if we fix L, M, N, and α there is no unique solution.

8.3.4. The maximal case. In contrast with the nonmaximal case, we could fix the shift α by asking for an extra vanishing moment for either the scaling or the wavelet function. (Because of the second remark in Remarks 4.3, an extra condition for the scaling function will actually add two vanishing moments.) In either case, there is at most a finite number of solutions or there are no solutions. If solutions exist they will be called *maximal* coiflets.

9. EXAMPLES

9.1. *Coiflets of Length 8*

We show how to construct all coiflets of length $L = 8$. In this case

$$\mathcal{V} = \{a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3\}.$$

The nonmaximal case. In order to obtain nonmaximal filters we choose $M = N = 3$ for our initial moments. From (8.4),

$$\begin{cases} a_0 = b_0 = \frac{1}{2}, \\ a_1 = a_2 = b_1 = b_2 = 0. \end{cases} \tag{9.1}$$

So, we only need to determine a_3, b_3, and α subject to (8.5). In this case there are only two equations:

$$a_4 + b_4 = 0 \qquad \text{and} \qquad a_6 + b_6 - {a_3}^2 - {b_3}^2 = 0.$$

As explained at the beginning of Section 8, using Part D of Lemma A.1 and (9.1) one can write a_4, b_4, a_6, and b_6 as linear combinations of a_3, b_3, and α.

The previous system then becomes

$$\begin{cases} -105 + 224\alpha - 130\alpha^2 + 28\alpha^3 - 2\alpha^4 + 1152a_3 - 384\alpha a_3 \\ \quad + 1536b_3 - 384\alpha b_3 = 0, \\ -1785 + 4312\alpha - 3703\alpha^2 + 1568\alpha^3 - 357\alpha^4 + 42\alpha^5 - 2\alpha^6 \\ \quad + 6912a_3 - 5760\alpha a_3 + 1728\alpha^2 a_3 - 192\alpha^3 a_3 - 9216{a_3}^2 \\ \quad + 14592b_3 - 9792\alpha b_3 + 2304\alpha^2 b_3 - 192\alpha^3 b_3 - 9216{b_3}^2 = 0. \end{cases} \tag{9.2}$$

An equivalent system, obtained via Gröbner bases where α is treated as parameter is

$$\begin{cases} 212625 - 599424\alpha + 704860\alpha^2 - 458360\alpha^3 + 181152\alpha^4 - 44624\alpha^5 \\ \quad + 6696\alpha^6 - 560\alpha^7 + 20\alpha^8 + (790272 - 558336\alpha + 10752\alpha^2 + 75264\alpha^3 \\ \quad - 19968\alpha^4 + 1536\alpha^5)a_3 + (3686400 - 2064384\alpha + 294912\alpha^2)a_3{}^2 = 0, \\ -105 + 224\alpha - 130\alpha^2 + 28\alpha^3 - 2\alpha^4 + (1152 - 384\alpha)a_3 \\ \quad + (1536 - 384\alpha)b_3 = 0. \end{cases} \tag{9.3}$$

The latter system helps in determining the range of values for α.

The left-hand side of the first equation in (9.3) is a polynomial of degree two in a_3. To have real solutions a_3 we require

$$\begin{aligned} &-29561 + 99568\alpha - 128100\alpha^2 + 87416\alpha^3 - 35448\alpha^4 + 8848\alpha^5 \\ &\quad - 1336\alpha^6 + 112\alpha^7 - 4\alpha^8 \geq 0. \end{aligned} \tag{9.4}$$

That is, α should belong to one of the following two intervals (approximate end points):

$$[0.681871, 3.09431] \qquad \text{or} \qquad [3.90568, 6.31812]. \tag{9.5}$$

Note that the intervals are symmetric about $\frac{7}{2}$, as discussed in Section 8.3.1. Also, in agreement with Proposition 3.2, both intervals are included in $(0, 7)$.

Only for α in these intervals can we solve (9.3) and therefore there is at least one coiflet with that particular shift α and three vanishing moments for both the wavelet and the scaling functions.

Coiflets with integer shifts. In this example, we can choose the shift to be any integer in the interval $(0, L-1)$. This is not possible in general as shown in Table 1.

Due to symmetry, we only consider $\alpha = 1, 2$, or 3. For instance, if $\alpha = 3$ in (9.3) then

$$a_3 = \pm\frac{\sqrt{7}}{128} \tag{9.6}$$

and

$$b_3 = \frac{3}{128}. \tag{9.7}$$

The two solutions for this case then have the following filter coefficients $\{h_0, \ldots, h_7\}$

$$\left\{-\frac{1}{32} - a_3, -\frac{3}{128}, \frac{9}{32} + 3\,a_3, \frac{73}{128}, \frac{9}{32} - 3\,a_3, -\frac{9}{128}, -\frac{1}{32} + a_3, \frac{3}{128}\right\}.$$

The choice of positive sign in (9.6) leads to the filter 3a, while the negative sign leads to filter 3b. Their numerical values are listed in Table 2. Note that because of (9.1) and (9.7), the polyphase component H_1 has rational coefficients.

The maximal case. From Sections 8.2 and 8.3.4, it follows that we can choose an extra moment for the wavelet function (by setting $a_3 = b_3$) or two extra moments for the scaling function (by setting $a_3 = -b_3$). In the latter case, (9.3) becomes

$$\begin{cases} 8505 - 36876\alpha + 66224\alpha^2 - 58576\alpha^3 + 28488\alpha^4 - 8008\alpha^5 \\ \quad + 1296\alpha^6 - 112\alpha^7 + 4\alpha^8 = 0, \\ 105 - 224\alpha + 130\alpha^2 - 28\alpha^3 + 2\alpha^4 + 384a_3 = 0. \end{cases} \tag{9.8}$$

Solving for real α in the first equation we obtain only two possible values in $(0, \frac{7}{2})$, namely,

$$\alpha_1 = 2.97727 \qquad \text{and} \qquad \alpha_2 = 2.23954. \tag{9.9}$$

Note that these values are in the first interval given in (9.5). The corresponding coefficients $\{h_k\}$ are listed in Table 2. Both filters have $M = 3$ and $N = 5$.

The extra moment for the wavelet will also lead to two solutions but with $M = 4$ and $N = 3$. They correspond to Daubechies' maximally flat filters of length 8. Their coefficients can be found in [8]. For coiflets of length $L > 10$, the number of vanishing moments of the scaling function is greater than three (this follows from (4.3) since $N \geq \frac{1}{3}(L-1)$). Therefore, these filters cannot coincide with Daubechies' family of filters.

Summary for filters of length 8. Within the region of possible shifts α, we found, up to symmetry, a total of six coiflets with integer shifts and four maximal coiflets. For these ten filters, -1 is the only root on the unit circle and therefore Cohen's condition is automatically satisfied. Nevertheless, their frequency responses are far from being uniform. A first distinction is related to the factorization (3.4).

In contrast with Daubechies' maximally flat filters, where $\|Q\| = \sup_{|z|=1} |Q(z)|$ is the same for all of them, $\|Q\|$ of different coiflets does indeed change.

When $\|Q\|$ is larger than 2^{M-1}, we can expect *bad* behavior for the filter and poor regularity for the associated scaling and wavelet functions. See [8, Lemma 7.1.1]. In Table 3, we listed the Sobolev exponents σ of the wavelet functions. They were computed using Theorem 9.5 in [20].

We have labeled our different solutions according to the size of $\|Q\|$. Thus, in Table 3, the filters Na and Nb correspond to the maximal case for the scaling function, but with $\|Q\| = 2.8764$ and $\|Q\| = 2.94511$, respectively.

We have labeled UGLY and BAD the cases for which $\|Q\|$ is increasingly larger than 2^{M-1}. Figures 3 and 4 shown this phenomena for the coiflet filters 2b and 1b (with integer shift $\alpha = 2$). Even for the *good* cases, where $\|Q\| < 2^{M-1}$, the filters exhibit a different behavior with respect to their phases.

Compare Figs. 5 and 6 for the filter 2a with Figs. 7 and 8 for the filter 3a. In the latter case, the phase of the filter has a sharp transition near π and therefore its group delay is much wider than for the case 2a. Nevertheless, since the module of the filter is zero at π that transition does not affect the overall response of the filter.

On the other hand, in agreement with Proposition 6.2 and because of the different number N of vanishing moments of the filters, the group delay for the case Na is flatter near zero than for the case with integer shifts. See Figs. 5, 7, and 9.

In Table 2 we listed the coefficients of filters of length 8 corresponding to maximal coiflets and coiflets with integer shifts. Because of (2.6), these coefficients of the low-pass filter m_0 sum to 1.

TABLE 2
Coiflet Filters of Length 8

	k	h_k		k	h_k
$M = 3$	0	−0.00899863735774892	$M = 3$	0	−0.03952785122359428
$N = 5$	1	−0.02054552466216258	$N = 5$	1	0.1271031281675352
$\alpha = \alpha_1$	2	0.2202099211463259	$\alpha = \alpha_2$	2	0.5323389066059403
Case Na	3	0.5701914465849665	Case Nb	3	0.440002251136967
MAXIMAL	4	0.3422577968313942	MAXIMAL	4	−0.005981694132267174
	5	−0.07306459213264614		5	−0.07120132136770919
	6	−0.05346908061997128		6	0.01317063874992116
	7	0.02341867020984207		7	0.004095942063206933
$M = 3$	0	0.1646660519380485	$M = 3$	0	0.3040839480619514
$N = 3$	1	0.5074101320413008	$N = 3$	1	0.414464867958699
$\alpha = 1$	2	0.4435018441858542	$\alpha = 1$	2	0.02524815581414562
Case 1a	3	−0.02223039612390291	Case 1b	3	0.2566053961239029
	4	−0.1310018441858543	BAD	4	0.2872518441858542
	5	0.02223039612390291		5	−0.2566053961239029
	6	0.02283394806195145		6	−0.1165839480619514
	7	−0.007410132041300974		7	0.085535132041301
$M = 3$	0	−0.01938529090153145	$M = 3$	0	0.0850102909015314
$N = 3$	1	0.1854738954507657	$N = 3$	1	0.1332761045492342
$\alpha = 2$	2	0.5581558727045942	$\alpha = 2$	2	0.2449691272954056
Case 2a	3	0.3810783136477028	Case 2b	3	0.5376716863522972
	4	−0.05815587270459436	UGLY	4	0.2550308727045943
	5	−0.06857831364770281		5	−0.2251716863522971
	6	0.01938529090153145		6	−0.0850102909015314
	7	0.002026104549234272		7	0.05422389545076572
$M = 3$	0	−0.05191993211769211	$M = 3$	0	−0.01058006788230788
$N = 3$	1	−0.0234375	$N = 3$	1	−0.0234375
$\alpha = 3$	2	0.3432597963530763	$\alpha = 3$	2	0.2192402036469236
Case 3a	3	0.5703125	Case 3b	3	0.5703125
	4	0.2192402036469236		4	0.3432597963530763
	5	−0.0703125		5	−0.0703125
	6	−0.01058006788230788		6	−0.05191993211769211
	7	0.0234375		7	0.0234375

Note. $\alpha_1 = 2.977273091796802$, $\alpha_2 = 2.239549738364678$.

9.2. Coiflets of Length 18

A similar analysis can be done for filters of length 18. In Table 4, we present a summary of our findings by listing the filter coefficients for two cases: coiflets with integers shifts and maximal coiflets. Filter coefficients are listed in Table 6.

Even at higher numbers of vanishing moments and different lengths, we still found UGLY and BAD filters. They always correspond to coiflets with integer shifts, but it is not a peculiarity of that case. Varying α, we found regions of nonmaximal coiflets with a similar behavior.

TABLE 3
Summary of All Maximal Coiflets and Coiflets with Integer Shifts for Length 8

Filter	α	M	N	σ	$\|Q\|$	2^{M-1}	Remarks
Na	2.97727	3	5	1.45584	2.8764	4	
Nb	2.23955	3	5	1.44599	2.94511	4	
Ma	1.00539	4	3	1.77557	5.91608	8	Daubechies' Extremal Phase
Mb	2.98547	4	3	1.77557	5.91608	8	Daubechies' Least Asymmetric
1a	1	3	3	1.77528	2.16403	4	
1b	1	3	3	0.14666	14.9356	4	BAD
2a	2	3	3	1.42232	3.11099	4	
2b	2	3	3	0.93596	6.91099	4	UGLY
3a	3	3	3	1.77341	2.16473	4	
3b	3	3	3	1.46353	2.82288	4	

Note. Coefficients are listed in Table 2. The maximal case for wavelets coincides with Daubechies' maximally flat filters.

In Figs. 11 and 12, we plotted $|m_0|$ and φ for the cases 6c (UGLY) and 5b (BAD) with length 18. The cases 7d and 6d, as listed in Table 4, exhibit a similar behavior. Even though their filter moduli do not oscillate as much as their counterparts of length 8, their behavior is clearly different than those for which $\|Q\|$ remains below 2^{M-1}. As an example of the latter situation, consider the filter 7c. The associated wavelet has only six vanishing moments, but its Sobolev exponent is higher than the exponent for Daubechies' wavelets which have nine vanishing moments.

Note that in all the plots for wavelets in the Fourier domain, the support of the functions is actually wider than shown.

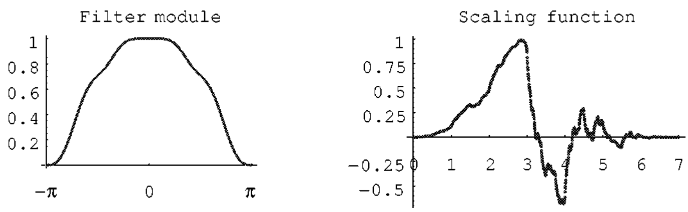

FIG. 3. Integer shift coiflet: length 8, shift 2, case b (UGLY). Plots of absolute value of filter m_0 and scaling function.

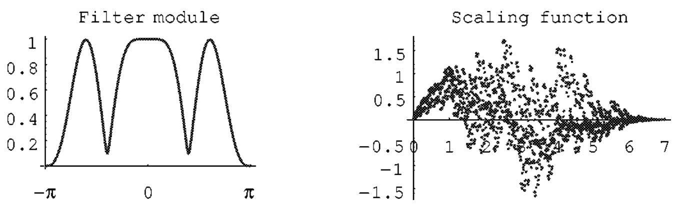

FIG. 4. Integer shift coiflet: length 8, shift 1, case b (BAD). Plots of absolute value of filter m_0 and scaling function.

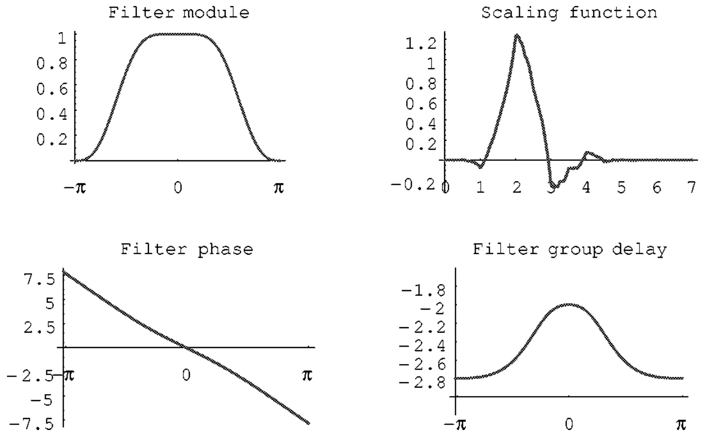

FIG. 5. Integer shift coiflet: length 8, shift 2, case a. Plots of scaling function and filter m_0.

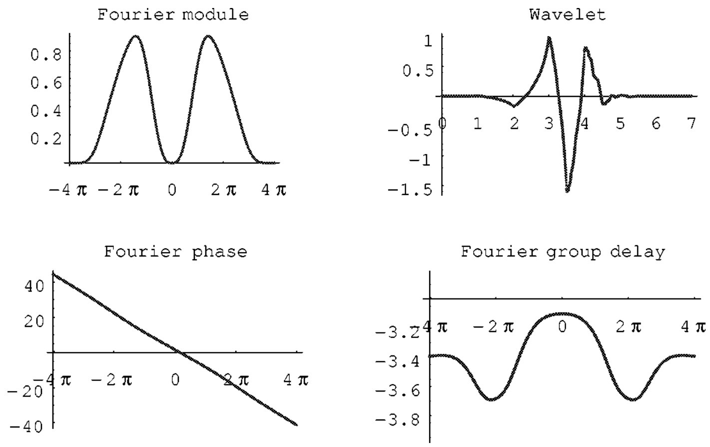

FIG. 6. Integer shift coiflet: length 8, shift 2, case a. Plots of wavelet function in both time and Fourier domain (absolute value, phase, and group delay).

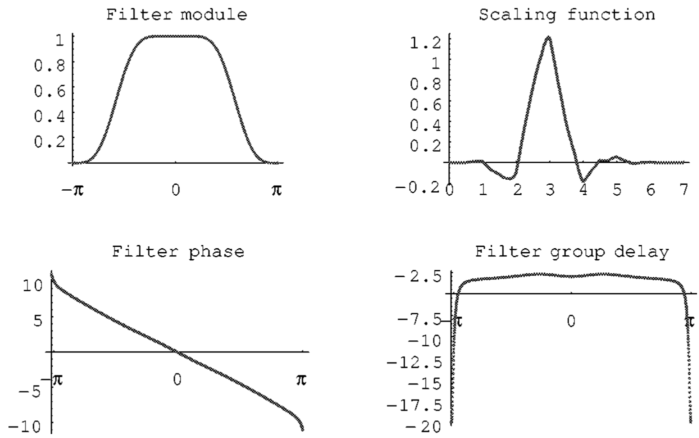

FIG. 7. Integer shift coiflet: length 8, shift 3, case a. Plots of scaling function and filter m_0.

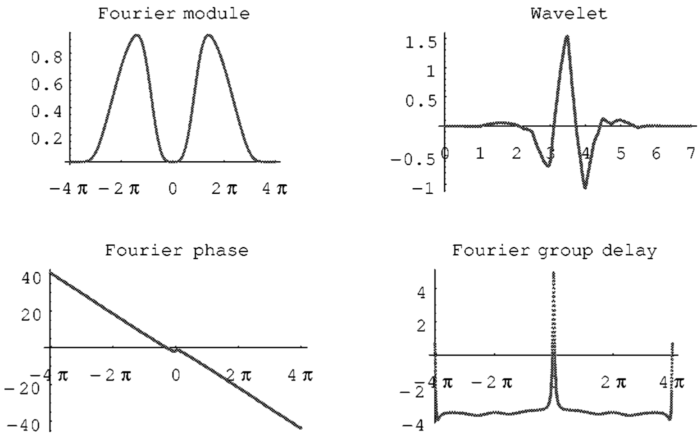

FIG. 8. Integer shift coiflet: length 8, shift 3, case a. Plots of wavelet function in both time and Fourier domain (absolute value, phase, and group delay).

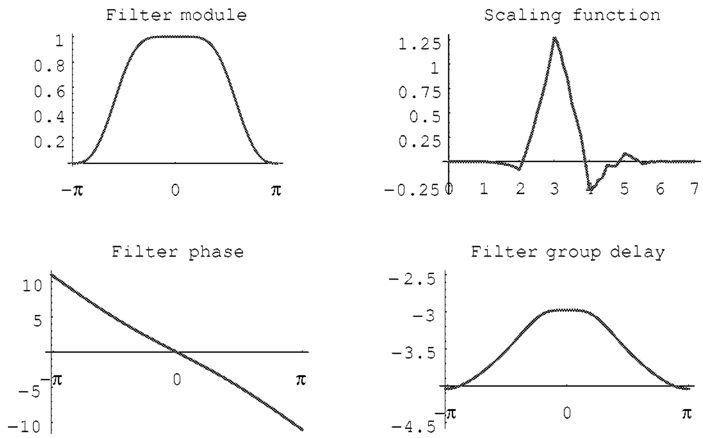

FIG. 9. Maximal coiflet for scaling function: length 8, shift 2.9773. Plots of the scaling function and filter m_0.

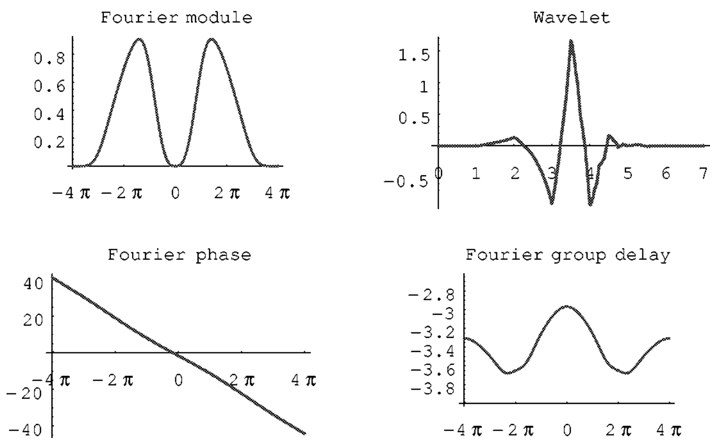

FIG. 10. Maximal coiflet for scaling function: length 8, shift 2.9773. Plots of the wavelet function in both time and Fourier domain (absolute value, phase, and group delay).

TABLE 4
Summary of All Maximal Coiflets, Coiflets with Integer Shifts, and Two Daubechies' Maximally Flat Filters for Length 18

Filter	α	M	N	σ	$\|Q\|$	2^{M-1}	Remarks
Na	7.81041	6	9	2.5149	16.5942	32	Listed in Table 5
Nb	7.1771	6	9	2.49853	17.2438	32	Listed in Table 5
Ma	5.94301	7	7	2.74543	33.9874	64	Listed in Table 5
Mb	4.5681	7	7	2.71944	36.2534	64	Listed in Table 5
5a	5	6	7	2.52726	15.3633	32	Listed in Table 6
5b	5	6	7	0.749459	99.1807	32	BAD
6a	6	6	7	2.73586	9.74416	32	
6b	6	6	7	2.48495	18.3793	32	Listed in [8, Table 8.1]
6c	6	6	7	1.89308	37.7778	32	UGLY
6d	6	6	7	0.697053	101.213	32	BAD
7a	7	6	7	2.59288	17.1479	32	
7b	7	6	7	2.46831	18.1119	32	Listed in Table 6
7c	7	6	7	3.29159	18.8021	32	
7d	7	6	7	1.77575	41.5161	32	UGLY
Dep	1.94435	9	3	3.16167	155.917	256	Daubechies' extremal phase
Dla	8.14657	9	3	3.16167	155.917	256	Daubechies' least asymmetric

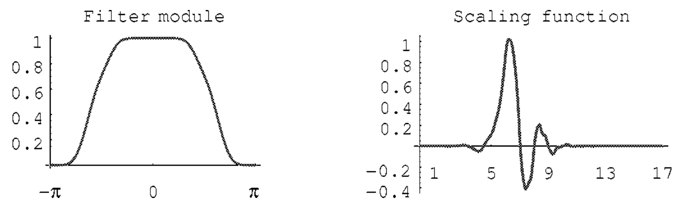

FIG. 11. Integer shift coiflet: length 18, shift 6, case c (UGLY). Plots of absolute value of filter m_0 and scaling function.

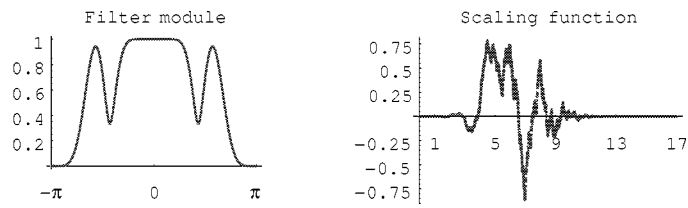

FIG. 12. Integer shift coiflet: length 18, shift 5, case b (BAD). Plots of absolute value of filter m_0 and scaling function.

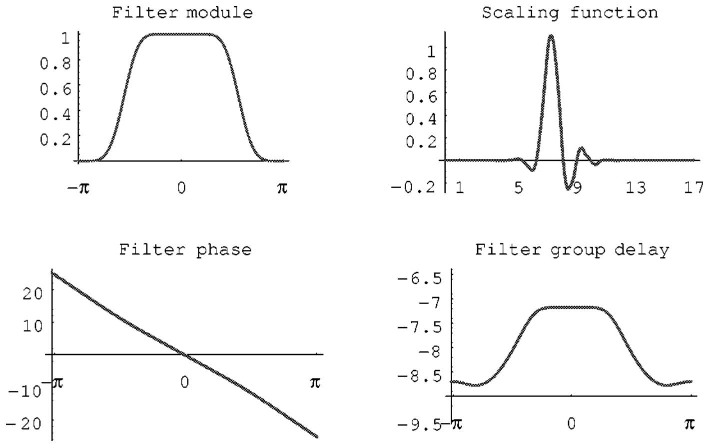

FIG. 13. Maximal coiflet for scaling function: length 18, shift 7.1771. Plots of filter m_0 and scaling function.

10. CONCLUSION

The approach taken in this paper allows one to construct and classify coiflets, which are wavelets with a high number of vanishing moments for both the scaling and wavelet functions. Coiflet filters are useful in applications where interpolation and linear phase are of particular importance.

We introduced a new system for FIR coiflets. In all cases investigated, the system had a minimal set of defining equations. For filters of length up to 20, the system can be solved explicitly, and the filter coefficients can thus be accurately determined. For longer filters we applied numerical methods to compute some solutions. For a few specific examples we studied the properties of coiflets corresponding to both integer and noninteger values of the first moment of the scaling function. Nevertheless, the problem of the existence of coiflet filters of arbitrary length and their full classification remains open.

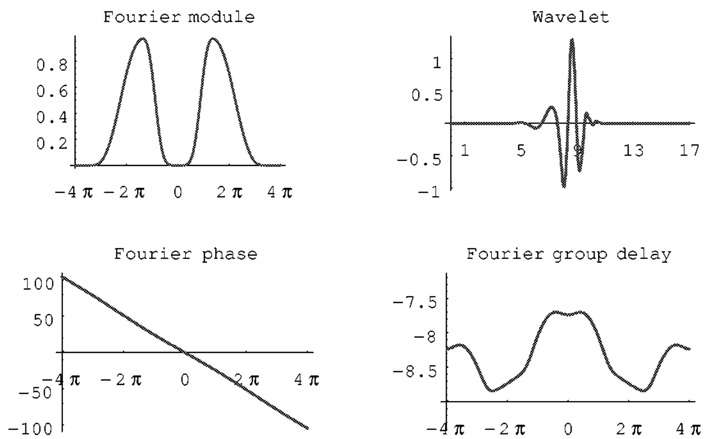

FIG. 14. Maximal coiflet for scaling function: length 18, shift 7.1771. Plots of wavelet function in both time and Fourier domain (absolute value, phase, and group delay).

TABLE 5
Coiflet Filters of Length 18: Maximal Case

	k	h_k		k	h_k
$M = 6$	0	−0.00006423105557385401	$M = 6$	0	−0.0002036914946771235
$N = 9$	1	−0.0002979447888413989	$N = 9$	1	−0.0002488151932121008
$\alpha = \alpha_1$	2	0.0004927238418624587	$\alpha = \alpha_2$	2	0.00221156402899935
Case Na	3	0.004159721116204626	Case Nb	3	0.005347581803838808
	4	−0.001356751057023208		4	−0.02049652597342785
	5	−0.03424128516618039		5	−0.03435328483085293
	6	0.01286924643513836		6	0.1757589722528208
	7	0.304174064910559		7	0.5137703862306729
	8	0.5487303262739295		8	0.4326537198943506
	9	0.2920015377606661		9	0.004003841371920543
	10	−0.0979310190825782		10	−0.1200187966274661
	11	−0.0822374057724846		11	0.02108432415813931
	12	0.05265614514287543		12	0.03561677266929025
	13	0.01690326579283296		13	−0.01228600681641712
	14	−0.01818476072132749		14	−0.005733199970056795
	15	0.0001391533251141822		15	0.002854859153956041
	16	0.002788320222696984		16	0.000211185220166843
	17	−0.0006011071778707536		17	−0.0001728858780453669
$M = 7$	0	0.0003232178738443985	$M = 7$	0	0.003401479882015607
$N = 7$	1	0.001666157023192355	$N = 7$	1	−0.004130806329954543
$\alpha = \tilde{\alpha}_1$	2	−0.001655740666688795	$\alpha = \tilde{\alpha}_2$	2	−0.03536170269249431
Case Ma	3	−0.02256218521490427	Case Mb	3	0.05747767104264993
	4	0.005072730487709637		4	0.3843902644404712
	5	0.2365835515640513		5	0.5358632409346619
	6	0.5195340737893435		6	0.1908760013178301
	7	0.3835397677855875		7	−0.1321131305836887
	8	−0.04580954371864931		8	−0.05295999083912471
	9	−0.1400028853157529		9	0.05813917906468963
	10	0.03870906867740069		10	0.009758111187504831
	11	0.05085645319997351		11	−0.01825628044991493
	12	−0.02266660403703607		12	0.0002608645070967113
	13	−0.0106114132773682		13	0.00327048515783943
	14	0.007588889762655687		14	−0.0003823627249285679
	15	0.0003179232674700494		15	−0.0002646325745805278
	16	−0.00109609216857971		16	0.000017334234085592
	17	0.0002126309677505884		17	0.00001427373829770887

Note. $\alpha_1 = 7.810413113222375$, $\alpha_2 = 7.177096173069426$, $\tilde{\alpha}_1 = 5.943011907827611$, $\tilde{\alpha}_2 = 4.568098992005785$.

APPENDIX

Assume that f and g are functions with enough derivatives, n is a nonnegative integer, and γ is a real constant.

D denotes the derivative operator and xD the operator $x\frac{d}{dx}$. For any operator T, T^0 is the identity operator. The nth iteration of xD is

TABLE 6
Coiflet Filters of Length 18: Two Integer Shifts

	k	h_k		k	h_k
$M = 6$	0	0.001440768926720368	$M = 6$	0	−0.0000629311510126045
$N = 7$	1	0.002053404421631864	$N = 7$	1	0.00004962145501794398
$\alpha = 5$	2	−0.02219838096076973	$\alpha = 7$	2	0.001740671204645141
Case 5a	3	−0.01250987368937947	Case 7b	3	0.001981652779610451
	4	0.2259647068843012		4	−0.02288745495628588
	5	0.5319491906628806		5	−0.01305004769565276
	6	0.3832103239740163		6	0.2273416538968731
	7	−0.04397844411169963		7	0.5339067763210922
	8	−0.1177139643780853		8	0.3805539932682246
	9	0.03476163022933876		9	−0.04661451530829168
	10	0.03446875381675335		10	−0.1140327606217869
	11	−0.01539096154107554		11	0.03625460316792878
	12	−0.005371847958435806		12	0.03137758456237431
	13	0.003361708092256614		13	−0.01539787454248928
	14	0.0002159461146890029		14	−0.003937649303352087
	15	−0.0002580954448262954		15	0.00298786428539753
	16	−0.00001630641918942108		16	−0.0000931068996797331
	17	0.00001144138087300107		17	−0.0001180804626132358

$$(xD)^n f(z) = \sum_{k=0}^{n} \mathcal{S}_k^n z^k D^k f(z), \tag{A.1}$$

where $\mathcal{S}_k^n$ are the Stirling numbers of the second kind.

These numbers have a closed-form given by

$$\mathcal{S}_k^n = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k+j} j^n. \tag{A.2}$$

The *falling factorial powers* of z are

$$z^{\underline{n}} = z(z-1)\cdots(z-(n-1)), \qquad z^{\underline{0}} = 1.$$

The change of basis relating $\{z^{\underline{n}}\}$ and $\{z^n\}$ is given in terms of the Stirling numbers of the first and second kind:

$$z^{\underline{n}} = \sum_{k=0}^{n} s_k^n z^k, \qquad z^n = \sum_{k=0}^{n} \mathcal{S}_k^n z^{\underline{k}}.$$

Therefore, s_k^n is the inverse matrix of $\mathcal{S}_k^n$ and then, for $n \le r$,

$$\sum_{i=k}^{r} \mathcal{S}_i^n s_k^i = \delta_{nk}. \tag{A.3}$$

From (A.1) and (A.3)

$$z^n D^n f(z) = \sum_{k=0}^{n} s_k^n (xD)^k f(z). \tag{A.4}$$

Note that for a polynomial of degree r, it is not true that $(xD)^n P(1)$ is zero for $n > r$. However, these values are linear combinations of $(xD)^n P(1)$ for $n \le r$, as we show in the next lemma.

LEMMA A.1. *For each k, n, r non-negative integers with $k \le r$, and γ any real number, define the polynomials*

$$L_{rk}(z) = \sum_{i=k}^{r} \frac{s_k^i}{i!}(z-1)^i \qquad \textit{and} \qquad L_{rk}^{\gamma}(z) = \sum_{i=k}^{r} i^{\underline{k}} \gamma^{i-k} L_{ri}(z).$$

Let P be any polynomial of at most degree r. We have the following properties:

$$\textbf{A} \qquad (xD)^n L_{rk}(1) = \sum_{i=k}^{r} \mathcal{S}_i^n s_k^i,$$

$$\textbf{B} \qquad P(z) = \sum_{k=0}^{r} (xD)^k P(1) L_{rk}(z),$$

$$\textbf{C} \qquad (xD)^n P(1) = \sum_{k=0}^{r} (xD)^k P(1) \sum_{i=k}^{r} \mathcal{S}_i^n s_k^i,$$

$$\textbf{D} \qquad P(z) = \sum_{k=0}^{r} \frac{(xD)^k (x^{-\gamma} P(x))(1)}{k!} L_{rk}^{\gamma}(z).$$

Proof. Part **A** follows from $(xD)^n \frac{(z-1)^i}{i!}(1) = \mathcal{S}_i^n$. To verify that, expand $(z-1)^i$ and use (A.2).

When $n \le r$, Part **A** and (A.3) imply that $(xD)^n L_{rk}(1) = \delta_{nk}$. Therefore, $\{L_{rk}\}_{k=0}^{r}$ are linearly independent and thus they are a basis for the polynomials of degree r or less. The representation of Part **B** then readily follows.

Part **C** is a consequence of Parts **A** and **B**.

By definition of L_{rk}^{γ}, the right-hand side of Part **D** equals

$$\begin{aligned}
&\sum_{k=0}^{r} (xD)^k (x^{-\gamma} P(x))(1) \sum_{i=k}^{r} \binom{i}{k} \gamma^{i-k} L_{ri}(z) \\
&= \sum_{i=0}^{r} \left(\sum_{k=0}^{i} \binom{i}{k} (xD)^{i-k} (x^{\gamma})(1) (xD)^k (x^{-\gamma} P(x))(1) \right) L_{ri}(z) \\
&= \sum_{i=0}^{r} (xD)^i (P)(1) L_{ri}(z).
\end{aligned}$$

Part **D** then follows using Part **B**. ■

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