APPM 4/5520

Section 7.1 Notes

Section 7.1 is about integration (yeah!), but there are no new integration techniques introduced. This section is about various tricks you can use to "smash" weird and scary looking integrands into familiar forms that you can handle with the techniques you have already learned.

Example 1: Completing the Square

Suppose we want to integrate

$$\int \frac{dx}{3x^2 - 6x + 8}$$

This would be really easy if I told you that

$$3x^2 - 6x + 8 = (\sqrt{5})^2 + [\sqrt{3}(x-1)]^2$$

because then the integrand has the form

$$\frac{du}{a^2 + u^2}$$

which integrates to

$$\frac{1}{a}\, tan^{-1}\left(\frac{u}{a}\right) + C.$$

Let's do the integral...

$$\int \frac{dx}{3x^2 - 6x + 8} = \int \frac{dx}{(\sqrt{5})^2 + [\sqrt{3}(x-1)]^2}$$

Let $u = \sqrt{3}(x-1)$. Then $du = \sqrt{3}\, dx$, so we rewrite the integral as

$$\frac{1}{\sqrt{3}} \int \frac{\sqrt{3}\, dx}{5^2 + [\sqrt{3}(x-1)]^2} = \frac{1}{\sqrt{3}} \int \frac{du}{(\sqrt{5})^2 + u^2}$$

$$= \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{5}}\, tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + C = \frac{1}{\sqrt{15}}\, tan^{-1}\left(\frac{\sqrt{3}(x-1)}{\sqrt{5}}\right) + C$$

That was no problem, but how did we get that $3x^2 - 6x + 8 = (\sqrt{5})^2 + [\sqrt{3}(x-1)]^2$?

We completed the square!

Completing the Square:

Step One:

Group together the first two terms and factor out the lead coefficient:

$$3x^2 - 6x + 8 = 3(x^2 - 2x) + 8.$$

(Note: If the quadratic expression were $-x^2 + 6x - 10$, for example, then the lead coefficient is a negative 1 and we would begin by writing $-(x^2 - 6x) - 10$.)

Step Two:

Take the remaining coefficient of the x-term, divide it by two, and square the result:

$$\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1.$$

Step Three:

Add the result of step two inside the parentheses and compensate for it by subtracting it on the outside of the parentheses:

$$3(x^2 - 2x + 1) + 8 - 3(1) = 3(x^2 - 2x + 1) + 5$$

(Note: We put a "1" inside the parentheses, but because of the "3" in front of the parentheses, we really added 3(1) into the expression. Therefore, we subtract 3(1) to balance things out.)

Step Four: Note that the stuff inside of the parentheses is a "perfect square" and enjoy!

$$3(x - 1)^2 + 5$$

(Note: You could see this by factoring $x^2 - 2x + 1$ into $(x - 1)(x - 1)$, but don't bother! Following this procedure, you will always get, as the squared term, an "x" followed by the number (including the sign) that you squared in step two above.)

Final Note on completing the square: Don't freak out if the letters change. You can follow all the steps above with a letter other than "x". I used the phrase "the 'x' term" so as not to complicate the explanation with language, but in general, for step two you take the coefficient of "the linear term" which is the term where the variable has an exponent of 1.

Example 2: Breaking Up a Fraction

Suppose we want to integrate

$$\int \frac{x+1}{3x^2-6x+8}\,dx.$$

You might start by letting $u = 3x^2 - 6x + 8$, but then $du = (6x-6)\,dx = 6(x-1)\,dx$ which is not some nice multiple of the numerator. So... this won't work. However, we can write

$$\frac{x+1}{3x^2-6x+8} = \frac{x-1+2}{3x^2-6x+8} = \frac{x-1}{3x^2-6x+8} + \frac{2}{3x^2-6x+8}$$

Now, integrating the first fraction is a "du over u" problem and the second integral is exactly (two times) the one we did back in Example 1! (Very cool!)

The first integral is

$$\int \frac{x-1}{3x^2-6x+8}\,dx = \frac{1}{6}\int \frac{6(x-1)}{3x^2-6x+8}\,dx = \frac{1}{6}\int \frac{du}{u}$$
$$= \frac{1}{6}\ \ln|u| + C = \frac{1}{6}\ \ln|3x^2-6x+8| + C.$$

The overall answer is

$$\int \frac{x+1}{3x^2-6x+8}\,dx = \frac{1}{6}\ \ln|3x^2-6x+8| + \frac{2}{\sqrt{15}}\,tan^{-1}\left(\frac{\sqrt{3}(x-1)}{\sqrt{5}}\right) + C.$$

Man, that looks intense!... but it really wasn't that bad was it? (Note: We combined the two "plus C's" into a single "plus C".)

Example 3: Dividing Out a Fraction

This technique may come in handy if the degree of the numerator is larger than the degree of the denominator.

For example, suppose we want to integrate

$$\int \frac{3x^2+1}{2x-3}\,dx.$$

We can use good old fashioned long division to simplify the integrand:

$$2x-3\,\overline{)\ 3x^2+1}$$

First, we must ask ourselves, "How many times does $2x$ go into $3x^2$?". That is "What times $2x$ will give us $3x^2$? The answer is $\frac{3}{2}x$ and so we write it on top:

$$\begin{array}{r l} & \frac{3}{2}x \\ \cline{2-2} 2x-3 \big) & 3x^2+1 \end{array}$$

Next, we multiply the $\frac{3}{2}x$ by the divisor $2x-3$ and subtract the result below:

$$\begin{array}{r l} & \frac{3}{2}x \\ \cline{2-2} 2x-3 \big) & 3x^2+1 \\ & -\left(3x^2-\frac{9}{2}x\right) \\ \cline{2-2} & \frac{9}{2}x+1 \end{array}$$

We now have a remainder of $\frac{9}{2}x+1$. We continue the procedure until the degree of the remainder is less than the degree of the divisor.

"How many times does $2x$ go into $\frac{9}{2}x$?" In other words, "What times $2x$ gives us $\frac{9}{2}x$?" The answer is $\frac{9}{4}$. So, we put a $\frac{9}{4}$ on top, multiply the divisor by $\frac{9}{4}$ and subtract the result:

$$\begin{array}{r l} & \frac{3}{2}x+\frac{9}{4} \\ \cline{2-2} 2x-3 \big) & 3x^2+1 \\ & -\left(3x^2-2x\right) \\ \cline{2-2} & \frac{9}{2}x+1 \\ & -\left(\frac{9}{2}x-\frac{27}{4}\right) \\ \cline{2-2} & \qquad\quad \frac{31}{4} \end{array}$$

So, we have that

$$\frac{3x^2+1}{2x-3} = \frac{3}{2}x + \frac{9}{4} + \frac{31/4}{2x-3}.$$

So,

$$\int \frac{3x^2+1}{2x-3}\,dx = \int \left[\frac{3}{2}x + \frac{9}{4} + \frac{31/4}{2x-3}\right] dx = \frac{3}{4}x^2 + \frac{9}{4}x + \frac{31}{8}\ln|2x-3| + C$$

(Also very cool!)

Example 4: Throwing Around Trig Identities

We've already done things like integrating $cos^2(x)$ by replacing it with $\frac{1}{2}[1 + cos(2x)]$, but have you ever considered putting in more trig functions than you started with in order to take advantage of an identity?

For example, suppose we want to integrate

$$\int sin(3x)csc(x)\,dx$$

We could use the fact that

$$sin(A + B) = sin(A)cos(B) + cos(A)sin(B).$$

Letting $A = x$ and $B = 2x$, the integral becomes

$$\int sin(3x)csc(x)\,dx = \int sin(x+2x)csc(x)\,dx = \int sin(x)cos(2x)csc(x)\,dx + \int cos(x)sin(2x)csc(x)\,dx$$

$$= \int cos(2x)\,dx + \int cos(x)sin(2x)csc(x)\,dx$$

Now using the fact that $sin(2x) = 2sin(x)cos(x)$, we have the integral as

$$= \int cos(2x)\,dx + 2\int cos(x)sin(x)cos(x)csc(x)\,dx$$

$$= \int cos(2x)\,dx + 2\int cos^2(x)\,dx$$

$$= \int cos(2x)\,dx + 2\int \frac{1}{2}[1 + cos(2x)]\,dx = \int cos(2x)\,dx + \int [1 + cos(2x)]\,dx$$

$$= \frac{1}{2}sin(2x) + x + \frac{1}{2}sin(2x)\,dx = sin(2x) + x$$

(Question: Can you reach the same result more quickly? Probably. Can you drag out the problem and make it much longer? Definitely! It is possible to do many things with trig identities, including setting yourself off in circles that get you nowhere. For this reason, integrating with trig functions can be challenging. It's a fun kind of challenging though– like a puzzle! Enjoy your trig integrals!)

(Note: Another common trig integral trick: If you have a fraction and you can't get away with a simple u-substitution, you might try multiplying the top and bottom of the fraction by a trig function that will then allow you to use identities. You might even try this without a fraction. For example

$$\int tan(x)\,dx = \int \frac{sec(x)tan(x)}{sec(x)} = \int \frac{du}{u}$$

where $u = sec(x)$. Therefore, the answer is

$$\ln|u| + C = \ln|sec(x)| + C.$$

Incidentally, this is the same thing as

$$-\ln|cos(x)| + C.$$

Woo hoo!)