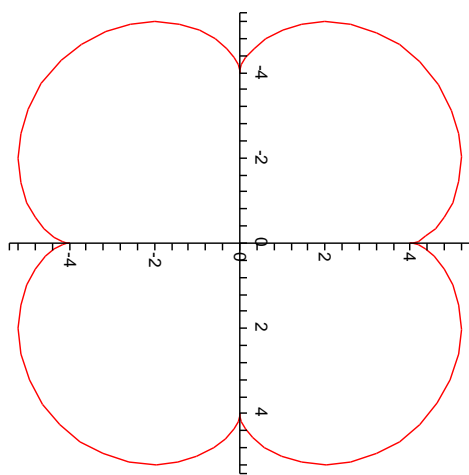


APPM 4/5520

Example from Class 04/18/05

Example: A circle of radius one rolls around the circumference of a larger circle of radius 4 centered at the origin. Let's follow one point on the smaller circle. The resulting curve traced out by this point is called an "epicycloid" and is depicted in Figure 1.

Figure 1: The Epicycloid Given by $x = 5 \cos t - \cos 5t$, $y = 5 \sin t - \sin 5t$



Our goal is to find the total length of this curve. When $t = 0$, we are at $(x, y) = (4, 0)$ on the curve. When $t = \pi/2$, we are at $(x, y) = (0, 4)$. Let's simplify the problem a bit by computing four times the length of that one part of the curve.

$$\begin{aligned}
 L &= 4 \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= 4 \int_0^{\pi/2} \sqrt{(-5 \sin t + 5 \sin 5t)^2 + (5 \cos t - 5 \cos 5t)^2} dt \\
 &= 4 \int_0^{\pi/2} \sqrt{25 \sin^2 t - 50 \sin t \sin 5t + 25 \sin^2 5t + 25 \cos^2 t - 50 \cos t \cos 5t + 25 \cos^2 5t} dt \\
 &= 4 \int_0^{\pi/2} \sqrt{25(\sin^2 t + \cos^2 t) + 25(\sin^2 5t + \cos^2 5t) - 50(\sin t \sin 5t + \cos t \cos 5t)} dt \\
 &= 4 \int_0^{\pi/2} \sqrt{25(1) + 25(1) - 50(\sin t \sin 5t + \cos t \cos 5t)} dt
 \end{aligned}$$

$$\begin{aligned} &= 4 \int_0^{\pi/2} \sqrt{50 - 50(\sin t \sin 5t + \cos t \cos 5t)} dt \\ &= 20 \int_0^{\pi/2} \sqrt{2 - 2(\sin t \sin 5t + \cos t \cos 5t)} dt \\ &= 20 \int_0^{\pi/2} \sqrt{2 - 2 \cos(5t - t)} dt \quad (\text{identity}) \\ &= 20 \int_0^{\pi/2} \sqrt{2(1 - \cos 4t)} dt \\ &= 20 \int_0^{\pi/2} \sqrt{4 \sin^2 2t} dt \quad (\text{identity}) \\ &= 40 \int_0^{\pi/2} \sin 2t dt \\ &= -20 \cos 2t \Big|_0^{\pi/2} = 40 \end{aligned}$$