APPM 4/5520

Solutions for Review Problems for Exam II, Sections 5.6-5.7

(Note: Some of these integrals were really hard with many steps, substitutions, and back-substitutions. Especially problems 1,2, and 3. If you can do these, great!! If not, don't be overly concerned. They turned out to be more difficult than I had intended for this review.)

1. Draw the picture, look at an " x-slice". Widen the slice just a little bit. Now you are looking at an approximate cylider. (The sides are not straight, they are given by a little piece of the sine curve). The radius of this "cylinder is given by $r = \sin x$, the height is approximatley given by the arc length integrand $h \approx \sqrt{1 + (f'(x))^2} = \sqrt{1 + \cos^2 x}$.

   The surface area of a cylinder is given by $2\pi rh$. Now we drag the cylindrical slice all the way from 0 to 2π:

   $$SA = \int_0^{2\pi} 2\pi \sin\, x \sqrt{1 + \cos^2\, x}\, dx$$

   $$\boxed{\textbf{THIS IS WRONG!}}$$

   This was tricky and it was not meant to be– don't worry, we wouldn't do this to you on the exam. If you do this integral (correctly!) you would get zero. A surface area of zero does not make any sense. The reason you would get zero is that $\sin\, x$ gets negative for $\pi < x < 2\pi$. So, in this region, the radius would be given by $r = 0 - \sin\, x = -\sin\, x$. Therefore, the integral should be done in two pieces:

   $$SA = \int_0^{\pi} 2\pi \sin\, x \sqrt{1 + \cos^2\, x}\, dx + \int_{\pi}^{2\pi} 2\pi(-\sin\, x) \sqrt{1 + \cos^2\, x}\, dx$$

   However, by symmetry, we should get the same surface area from the surface of revolution generated by the 0 to π part and that generated by the π to 2π part. So, we will evaluate the 0 to π part and multiply by 2:

   $$SA = 4\pi \int_0^{\pi} \sin\, x \sqrt{1 + \cos^2\, x}\, dx$$

   Now to do the integral:

   Be careful not to make the mistake of saying that $1 + \cos^2\, x$ is $\sin^2\, x$, because it's not! ($\sin^2\, x = \cos^2\, x = 1 \;\Rightarrow\; \sin^2\, x = 1 - \cos^2\, x$) Hmmm... we'll need another approach.

   Let's try letting $u = \cos x$. This is a good start since it's derivative can be found in the integrand. $du = -\sin\, x\, dx$. The integral becomes

   $$SA = -4\pi \int_0^{\pi} \sqrt{1 + \cos^2\, x}\, (-\sin\, x)\, dx = -4\pi \int_1^{-1} \sqrt{1 + u^2}\, du = 4\pi \int_{-1}^{1} \sqrt{1 + u^2}\, du$$

Now this looks like a "trig substitution integral". We will let $u = \tan\,\theta$. Then $du = \sec^2\,\theta$ and $1 + u^2 = 1 + \tan^2\,\theta = \sec^2\,\theta$. Thus, the integral becomes

$$SA = 4\pi \int_{-\pi/4}^{\pi/4} \sqrt{\sec^2\,\theta}\,\sec^2\theta\,d\theta = 4\pi \int_{-\pi/4}^{\pi/4} \sec^3\theta\,d\theta$$

(Note that we changed the limits of integration again with this second change of variables. You could continue through the problem without the change, put the x's back in the end, and use the original limits 0 and π. Just don't write the 0 to π on the "u-integral" or the "θ-integral"– they are limits for x and don't belong with the u and θ!)

We will now integrate $\sec^3\,\theta$ without worrying yet about the limits of integration and the constant 4π.

Since we know how to take the derivative of $\sec\,\theta$ and how to integrate $\sec^2\,\theta$, we can try integration by parts ($\int u\,dv = uv - \int v\,du$). Let $u = \sec\,\theta$ and $dv = \sec^2\,\theta\,d\theta$. Then $du = \sec\,\theta\,\tan\,\theta\,d\theta$ and $v = \tan\,\theta$. So, the integral becomes:

$$\int \sec^3\,x\,dx = \int \sec\,x \cdot \sec^2\,x\,dx$$

$$= \sec\,\theta\,\tan\,\theta - \int \sec\,\theta\,\tan^2\,\theta\,d\theta$$

Using the trig identity $\sec^2\,\theta = 1 + \tan^2\,\theta$, we may replace the $\tan^2\,\theta$:

$$\begin{aligned} \textstyle\int \sec^3\,\theta\,d\theta &= \sec\,\theta\,\tan\,\theta - \textstyle\int \sec\,\theta\,(\sec^2\,\theta - 1)\,d\theta \\ &= \sec\,\theta\,\tan\,\theta - \textstyle\int \sec^3\,\theta\,d\theta + \textstyle\int \sec\,\theta\,d\theta \end{aligned}$$

So, moving one integral from the right hand side to the left:

$$2\int \sec^3\,\theta\,d\theta = \sec\,\theta\,\tan\,\theta + \int \sec\,\theta\,d\theta$$

$$= \sec\,\theta\,\tan\,\theta + \int \sec\,\theta \cdot \frac{\sec\,\theta + \tan\,\theta}{\sec\,\theta + \tan\,\theta}\,d\theta$$

$$= \sec\,\theta\,\tan\,\theta + \int \frac{\sec^2\,\theta + \sec\,\theta\tan\,\theta}{\sec\,\theta + \tan\,\theta}\,d\theta$$

Letting $w = \sec\,\theta + \tan\,\theta$, we have $dw = \sec^2\,\theta + \sec\,\theta\tan\,\theta$. So the second integral is of the form dw/w which gives us natural log:

$$2\int \sec^3\,\theta\,d\theta = \sec\,\theta\,\tan\,\theta + \ln|\sec\,\theta + \tan\,\theta| + C$$

So

$$\int \sec^3\,\theta\,d\theta = \frac{1}{2}\sec\,\theta\,\tan\,\theta + \frac{1}{2}\ln|\sec\,\theta + \tan\,\theta| + C.$$

So, the final answer for this problem is

$$
\begin{aligned}
SA = 4\pi \int_{-\pi/4}^{\pi/4} \sec^3\, d\theta &= 2\pi \left[\sec\,\theta\,\tan\,\theta + \ln|\sec\,\theta + \tan\,\theta|\right]_{-\pi/4}^{\pi/4} \\
&= 2\pi \left[\left(\sqrt{2}(1) + \ln|\sqrt{2}+1|\right) - \left(\sqrt{2}(-1) + \ln|\sqrt{2} + (-1)|\right)\right] \\
&= 2\pi \left[2\sqrt{2} + \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]
\end{aligned}
$$

---

2. (Change this problem to a "set up but do not integrate problem".)

   See the set up for problem 1 above. The resulting integral here is

   $$SA = 2\pi \int_0^1 x^2\sqrt{1+(2x)^2}\, dx$$

   I did this in a couple of ways– one involving integration by parts that had to be followed by a trig substitution, another involving trig substitution followed by integration by parts. In both cases I had to repeat the process at least once and use a trick similar to the one in the solution above where I subtracted an integral from both sides. It was very ugly and will take many pages for me to type.

   If you care about the answer, it is:

   $$
   \begin{aligned}
   &2\pi\left[\frac{1}{16}(1+4x^2)^{3/2} - \frac{1}{32}(1+4x^2)^{1/2} - \frac{1}{64}\sin^{-1}(2x)\right]_0^1 \\
   &= -\frac{1}{32}\pi\left[-18\sqrt{5} + \ln(\sqrt{5}+2)\right]
   \end{aligned}
   $$

   Don't use your precious study time trying to compute this. Just make sure you can set up the integral for this surface area problem and do all the integrals in the remaining review problem sets. (Those problems were plucked from the text, so they've already been tested as "reasonable" problems.)

---

3. (Change this problem to a "set up but do not integrate problem".)

   Check out the more detailed set-up for the solution to problem 1 above. Here, we are spinning out a cylinder with radius $r = \ln x - (-1) = \ln x + 1$ and "height" $h \approx \sqrt{1+\left(\frac{d}{dx}\ln x\right)^2} = \sqrt{1+\left(\frac{1}{x}\right)^2}$. The integral is

   $$SA = 2\pi\int_1^e (\ln x + 1)\sqrt{1+\left(\frac{1}{x}\right)^2}\, dx = 2\pi\int_1^e (\ln x+1)\sqrt{\frac{x^2+1}{x^2}}\, dx$$

$$= 2\pi \int_1^e \frac{(\ln x + 1)}{x} \sqrt{x^2+1}\, dx$$

(Note that $\sqrt{x^2} = |x| = x$ here since x is positive. (It is in between 1 and e.)

---

4. Here, we are spinning out a cylinder with radius $r = \sqrt{4y - y^2}$ and "height" $h \approx \sqrt{1 + \left(\frac{d}{dy}\sqrt{4y-y^2}\right)^2} = \sqrt{1 + \left(\frac{2-y}{\sqrt{4y-y^2}}\right)^2}$. The integral is

$$SA = 2\pi \int_1^2 \sqrt{4y-y^2}\left(1 + \frac{(2-y)^2}{4y-y^2}\right) dy$$

$$= 2\pi \int_1^2 \sqrt{4y-y^2}\left(\frac{4}{4y-y^2}\right) dy$$

$$= 8\pi \int_1^2 \frac{1}{\sqrt{4y-y^2}}\, dy = 8\pi \sin^{-1}(x/2-1)\Big|_1^2 = \frac{4\pi^2}{3}$$

---

5. Take an x-strip, make it a little "fatter"– that is, widen it so that it is a rectangle with a small width called dx. The height of the rectangle is $2\sin(2x)$, so the area is

$$dA = 2\sin(2x)\, dx.$$

Since mass is density times area, we have that the mass of the strip is

$$dm = \delta \cdot dm = 2\delta \sin(2x)\, dx.$$

Call the center of mass for this little rectangular strip $(\tilde{x}, \tilde{y})$. It is in the center of the rectangle and therefore:

$$\tilde{x} = x, \qquad \text{and} \qquad \tilde{y} = \sin(2x).$$

We got the y-coordinate of the rectangle by averaging the top and bottom:

$$\tilde{y} = \frac{2\sin(2x) + 0}{2} = \sin(2x).$$

The total mass of the system is

$$M = \int dm = \int_0^{\pi/2} 2\delta \sin(2x)\, dx = 2\delta \int_0^{\pi/2} \sin(2x)\, dx$$

$$= 2\delta \left[-\frac{1}{2}\cos(2x) \right]_0^{\pi/2} = -\delta[cos(\pi) - \cos(0)] = -\delta[-1-1] = 2\delta.$$

The moment of the system about the x-axis is

$$M_x = \int \tilde{y}\, dm = \int_0^{\pi/2} \sin(2x) \cdot 2\delta \sin(2x)\, dx$$

$$= 2\delta \int_0^{\pi/2} \sin^2(2x)\, dx = 2\delta \int_0^{\pi/2} \frac{1-\cos(4x)}{2}\, dx$$

$$= \delta \int_0^{\pi/2} (1 - \cos(4x))\, dx = \delta \left[x - \frac{1}{4}\sin(4x) \right]_0^{\pi/2}$$

$$= \delta \left[\left(\frac{\pi}{2} - \frac{1}{4}\sin(2\pi) \right) - (0-0) \right] = \frac{\pi}{2}\delta.$$

The moment of the system about the y-axis is

$$M_y = \int \tilde{x}\, dm = \int_0^{\pi/2} x \cdot 2\delta \sin(2x)\, dx = 2\delta \int_0^{\pi/2} x \cdot \sin(2x)\, dx$$

Use integration by parts with $u = x$ and $dv = \sin(2x)\, dx$:

$$= 2\delta \left[\frac{1}{4}\sin(2x) - \frac{1}{2}x\cos(2x) \right]_0^{\pi/2} = 2\delta \cdot \frac{\pi}{4} = \frac{\pi}{2}\delta.$$

So, the x-coordinate of the center of mass is

$$\overline{x} = \frac{M_y}{M} = \frac{(\pi/2)\delta}{2\delta} = \frac{\pi}{4}.$$

The y-coordinate of the center of mass is

$$\overline{y} = \frac{M_x}{M} = \frac{(\pi/2)\delta}{2\delta} = \frac{\pi}{4}.$$

The center of mass is

$$\left(\frac{\pi}{4}, \frac{\pi}{4} \right).$$

---

6. These curve intesect at $x = 0$ and $x = 1$. In this region, $\sqrt{x}$ is above x^3.

   Take an x-strip, make it a little "fatter"– that is, widen it so that it is a rectangle with a small width called dx. The height of the rectangle is $\sqrt{x} - x^3$, so the area is

   $$dA = (\sqrt{x} - x^3)\, dx.$$

Since mass is density times area, we have that the mass of the strip is

$$dm = \delta(\sqrt{x} - x^3)\, dx.$$

Call the center of mass for this little rectangular strip $(\tilde{x}, \tilde{y})$. It is in the center of the rectangle and therefore:

$$\tilde{x} = x, \qquad \text{and} \qquad \tilde{y} = \frac{\sqrt{x} + x^3}{2}.$$

(We got the y-coordinate of the rectangle by averaging the top and bottom.)

The total mass of the system is

$$M = \int dm = \delta \int_0^1 (\sqrt{x} - x^3)\, dx = \frac{5}{12}\delta.$$

The moment of the system about the x-axis is

$$M_x = \int \tilde{y}\, dm = \int_0^1 \frac{\sqrt{x} + x^3}{2} \cdot \delta(\sqrt{x} - x^3)\, dx$$

$$= \frac{\delta}{2} \int_0^1 (\sqrt{x} + x^3)(\sqrt{x} - x^3)\, dx = \frac{\delta}{2} \int_0^1 (x - x^6)\, dx = \frac{5}{28}\delta.$$

The moment of the system about the y-axis is

$$M_y = \int \tilde{x}\, dm = \int_0^1 x \cdot \delta(\sqrt{x} - x^3)\, dx = \delta \int_0^1 (x^{3/2} - x^4)\, dx = \frac{1}{5}\delta.$$

So, the x-coordinate of the center of mass is

$$\overline{x} = \frac{M_y}{M} = \frac{(1/5)\delta}{(5/12)\delta} = \frac{1}{12}\delta.$$

The y-coordinate of the center of mass is

$$\overline{y} = \frac{M_x}{M} = \frac{(5/28)\delta}{(5/12)\delta} = \frac{12}{28}\delta.$$

The center of mass is

$$\left(\frac{1}{12}, \frac{12}{28}\right).$$

---

7. Take an x-strip, make it a little "fatter"– that is, widen it so that it is a rectangle with a small width called dx. The height of the rectangle is $(4/\sqrt{x}) - (-4/\sqrt{x}) = 8/\sqrt{x}$, so the area is

$$dA = \frac{8}{\sqrt{x}}\,dx.$$

Since mass is density times area, we have that the mass of the strip is

$$dm = \delta \frac{8}{\sqrt{x}}\,dx = \frac{1}{x} \cdot \frac{8}{\sqrt{x}}\,dx = \frac{8}{x^{3/2}}\,dx = 8x^{-3/2}\,dx.$$

Call the center of mass for this little rectangular strip $(\tilde{x}, \tilde{y})$. It is in the center of the rectangle and therefore:

$$\tilde{x} = x, \qquad \text{and} \qquad \tilde{y} = \frac{(4/\sqrt{x}) + (-4/\sqrt{x})}{2} = 0.$$

(We got the y-coordinate of the rectangle by averaging the top and bottom.)

The total mass of the system is

$$M = \int dm = \int_1^4 8x^{-3/2}\,dx = 8.$$

The moment of the system about the x-axis is

$$M_x = \int \tilde{y}\,dm = \int_1^4 0 \cdot 8x^{-3/2}\,dx = 0$$

The moment of the system about the y-axis is

$$M_y = \int \tilde{x}\,dm = \int_1^4 x \cdot 8x^{-3/2}\,dx = 8\int_1^4 x^{-1/2}\,dx = 16.$$

So, the x-coordinate of the center of mass is

$$\overline{x} = \frac{M_y}{M} = \frac{16}{8} = 2.$$

The y-coordinate of the center of mass is

$$\overline{y} = \frac{M_x}{M} = \frac{0}{8} = 0.$$

The center of mass is

$$(2, 0)\,.$$

---

8. Okay, if you sketch this out, you'll see that the curve $y = -\sqrt{x}$ is irrelevant to the problem. There is only a region bounded between $y = \sqrt{x}$ and $y = x/2$. (This was unintentional– we are not out to "trick" you on the exam!)

   Since the density is given in terms of y, we must translate everything else to y. That is, we are considering the area bounded by the curves $x = y^2$ and $x = 2y$. These curves intersect at $y = 0$ and $y = 2$.

   Take a y-strip, make it a little "fatter"– that is, widen it so that it is a rectangle with a small width called dy. The length of the rectangle is $2y - y^2$, so the area is

$$dA = (2y - y^2)\,dy.$$

   Since mass is density times area, we have that the mass of the strip is

$$dm = \delta(2y - y^2)\,dy = (y+1)(2y - y^2)\,dy = (y^2 - y^3 + 2y)\,dy.$$

   Call the center of mass for this little rectangular strip $(\tilde{x}, \tilde{y})$. It is in the center of the rectangle and therefore:

$$\tilde{x} = \frac{2y + y^2}{2}, \qquad \text{and} \qquad \tilde{y} = y.$$

   (We got the x-coordinate of the rectangle by averaging the right and left.)

   The total mass of the system is

$$M = \int dm = \int_0^2 (y^2 - y^3 + 2y)\,dy = \frac{8}{3}.$$

   The moment of the system about the x-axis is

$$M_x = \int \tilde{y}\,dm = \int_0^2 y \cdot (y^2 - y^3 + 2y)\,dy = \frac{44}{15}$$

   The moment of the system about the y-axis is

$$M_y = \int \tilde{x}\,dm = \int_0^2 \frac{2y + y^2}{2} \cdot (y^2 - y^3 + 2y)\,dy = \frac{24}{5}.$$

   (To integrate that, multiply it all out and integrate term by term.)

   So, the x-coordinate of the center of mass is

$$\overline{x} = \frac{M_y}{M} = \frac{24/5}{8/3} = \frac{9}{5}.$$

   The y-coordinate of the center of mass is

$$\overline{y} = \frac{M_x}{M} = \frac{44/15}{8/3} = \frac{11}{10}.$$

   The center of mass is

$$\left(\frac{9}{5}, \frac{11}{10}\right).$$