

APPM 4/5520

Solutions for Review Problems for Exam II, Sections 7.1-7.2

11. Integration by parts with $u = \sin^{-1} x$ and $dv = dx$. Then $du = \frac{1}{\sqrt{1-x^2}} dx$ and $v = x$.

Then

$$\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

The second integral can be done with a simple u -substitution: $u = 1 - x^2$. The final answer is

$$\int \sin^{-1} x dx = x \sin^{-1} x - \sqrt{1-x^2} + C$$

12. Let $u = \ln x$ and $dv = x^3 dx$. Then $du = \frac{1}{x} dx$ and $v = \frac{1}{4}x^4$.

We now have

$$\begin{aligned} \int x^3 \ln x dx &= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x} dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4} \int x^3 dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C \end{aligned}$$

So,

$$\int_1^e x^3 \ln x dx = \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 \right]_1^e = \left(\frac{1}{4}e^4 - \frac{1}{16}e^4 \right) - \left(0 - \frac{1}{16} \right) = \frac{1}{16}(3e^4 + 1)$$

- 13.

$$\begin{aligned} \int \csc x dx &= \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x} dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx \\ &= - \int \frac{-\csc^2 x - \csc x \cot x}{\csc x + \cot x} dx = - \int \frac{du}{u} \end{aligned}$$

where $u = \csc x + \cot x$.

Therefore we conclude that

$$\int \csc x dx = - \ln |\csc x + \cot x| + C$$

14. Let $u = \ln x + 2$. Then $du = \frac{1}{x} dx$ and the integral becomes

$$\int \cot u du = \int \cot u \frac{\csc u}{\csc u} du$$

Let $v = \csc u$. Then $dv = -\csc u \cot u du$ and the integral becomes

$$- \int \frac{dv}{v} = - \ln |v| = - \ln |\csc u| + C = - \ln |\csc(\ln x + 2)| + C.$$

(Incidentally, this can also be written as $\ln |\sin(\ln x + 2)| + C$.)

15.

$$\int_{-\pi/4}^{\pi/4} \sqrt{1 + \tan^2 w} dw = \int_{-\pi/4}^{\pi/4} \sqrt{\sec^2 w} dw = \int_{-\pi/4}^{\pi/4} |\sec w| dw = \int_{-\pi/4}^{\pi/4} \sec w dw$$

since secant is positive on the interval from $-\pi/4$ to $\pi/4$.

Multiplying top and bottom by $\sec w + \tan w$, we end up with

$$\ln |\sec w + \tan w| \Big|_{-\pi/4}^{\pi/4} = \ln |\sqrt{2} + 1| - \ln |\sqrt{2} - 1| = \ln \left| \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right| = \ln \left[\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right]$$

16. This is one of those “integration by parts twice and then collect the integrals on one side” problems.

Let $u = e^{-x}$ and $dv = \cos x dx$. Then $du = -e^{-x} dx$ and $v = \sin x$. So

$$\int e^{-x} \cos x dx = x \sin x + \int e^{-x} \sin x dx$$

Now for the last integral, we again do integration by parts, letting $u = e^{-x}$ and $dv = \sin x dx$. Then $du = -e^{-x} dx$ and $v = -\cos x$ and we have

$$\int e^{-x} \cos x dx = x \sin x + [-e^{-x} \cos x - \int e^{-x} \cos x dx]$$

Collecting the “target integral” on the left hand side, we get

$$2 \int e^{-x} \cos x dx = x \sin x - e^{-x} \cos x$$

So

$$\int e^{-x} \cos x dx = \frac{1}{2} x \sin x - \frac{1}{2} e^{-x} \cos x + C$$

17. Integration by parts with $u = \tan^{-1} x$ and $dv = x dx$. Then $du = \frac{1}{1+x^2} dx$ and $v = x^2/2$. So, we have

$$\int x \tan^{-1} x dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

Long division of the last integrand gives $1 - \frac{1}{1+x^2}$, so after integrating each of these, we end up with

$$\int x \tan^{-1} x dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + C$$

18. Integration by parts with $u = x^3$ and $dv = 3^x dx$. Then $du = 3x^2 dx$ and $v = \frac{1}{\ln 3} 3^x$ and we have

$$\int x^3 3^x dx = \frac{1}{\ln 3} x^3 3^x - \frac{3}{\ln 3} \int x^2 3^x dx$$

Now do integration by parts again to compute that last integral by letting $u = x^2$ and $dv = 3^x dx$.

After that, you will have to do integration by parts one more time. The final answer is:

$$\frac{3^x}{(\ln 3)^4} [x^3 (\ln 3)^3 - 3x^2 (\ln 3)^2 + 6x \ln 3 - 6]$$

19. Integration by parts with $u = \ln x$ and $dv = dx$. Then $du = (1/x) dx$ and $v = x$. So, we get

$$\int \ln x dx = x \ln x - \int \frac{1}{x} \cdot x dx = x \ln x - \int dx = x \ln x - x + C$$

So,

$$\int_1^e \ln x dx = [x \ln x - x]_1^e = (e - e) - (0 - 1) = 1$$

20. Integration by parts with $u = (\ln x)^2$ and $dv = x^{-2} dx$ gives $du = 2 \ln x \cdot \frac{1}{x} dx$ and $v = -1/x$. So,

$$\int (\ln x)^2 \cdot x^{-2} dx = -\frac{(\ln x)^2}{x} + 2 \int \frac{\ln x}{x^2} dx$$

A second integration by parts with $u = \ln x$ and $dv = (1/x^2) dx$ gives $du = (1/x) dx$ and $v = -1/x$. So

$$\begin{aligned} \int (\ln x)^2 \cdot x^{-2} dx &= -\frac{(\ln x)^2}{x} + 2 \left[-\frac{\ln x}{x} + \int \frac{1}{x^2} dx \right] \\ &= -\frac{(\ln x)^2}{x} - 2\frac{\ln x}{x} + \frac{2}{x} + C \end{aligned}$$