

APPM 4/5520

Solutions for Review Problems for Exam II, Sections 7.3-7.4

1. The denominator factors into $(x - 3)(x - 4)$. A partial fractions decomposition gives

$$\frac{2x + 1}{x^2 - 7x + 12} = \frac{-7}{x - 3} + \frac{9}{x - 4}$$

So,

$$\int \frac{2x + 1}{x^2 - 7x + 12} dx = \int \frac{9}{x - 4} dx - \int \frac{7}{x - 3} dx = 9 \ln |x - 4| - 7 \ln |x - 3| + C$$

(This can also be written as $\ln \left| \frac{(x-4)^9}{x-3} \right| + C$.)

2. $y^2 + 9$ is not reducible, so a partial fractions decomposition is

$$\frac{y^4 + 81}{y(y^2 + 9)^2} = \frac{A}{y} + \frac{By + C}{y^2 + 9} + \frac{Dy + E}{(y^2 + 9)^2}$$

Solving this gives $A = 1$, $B = 0$, $C = 0$, $D = -18$, and $E = 0$. So,

$$\frac{y^4 + 81}{y(y^2 + 9)^2} = \frac{1}{y} - \frac{18y}{(y^2 + 9)^2}$$

Integrating the first one gives $\ln |y|$. The second one is a simple u -substitution $u = y^2 + 9$. The final answer is

$$\int \frac{y^4 + 81}{y(y^2 + 9)^2} dy = \ln |y| + \frac{9}{y^2 + 9} + C$$

3. This is an improper fraction– the degree of the numerator is larger than the degree of the denominator. Before we do anything, we must perform the long division:

$$\frac{2y^4}{y^3 - y^2 + y - 1} = 2y + 2 + \frac{2}{y^3 - y^2 + y - 1}$$

That last term can be rewritten as

$$\frac{2}{(y^2 + 1)(y - 1)} \stackrel{\text{part. frac.}}{=} \frac{1}{y - 1} - \frac{y + 1}{y^2 + 1}$$

So

$$\begin{aligned} \int \frac{2y^4}{y^3 - y^2 + y - 1} dy &= 2 \int (y + 1) dy + \int \frac{1}{y - 1} dy - \int \frac{y}{y^2 + 1} dy - \int \frac{1}{y^2 + 1} dy \\ &= y^2 + 2y + \ln |y - 1| - \frac{1}{2} \ln |y^2 + 1| - \tan^{-1}(y) + C \end{aligned}$$

4. Let $x = \sin \theta$. Then $dx = \cos \theta d\theta$ and $1 - x^2 = 1 - \sin^2 \theta = \cos^2 \theta$ and the integral becomes

$$\begin{aligned} \int \frac{(\cos^2 \theta)^{3/2}}{\sin^6 \theta} \cos \theta d\theta &= \int \frac{(\cos \theta)^3}{\sin^6 \theta} \cos \theta d\theta = \int \frac{\cos^4 \theta}{\sin^6 \theta} d\theta \\ &= \int \cot^4 \theta \csc^2 \theta d\theta \end{aligned}$$

Since the derivative of cotangent is negative cosecant squared, let $u = \cot \theta$, then $du = -\csc^2 \theta$ and the integral becomes

$$-\int u^4 du = -\frac{1}{5}u^5 + C = -\frac{1}{5}\cot^5 \theta + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$$

5. This is a separable equation:

$$\frac{1}{3} dy = \frac{dx}{x^2 + 4}$$

(We could have left the 3 on the right side.)

Integrating both sides:

$$\int \frac{1}{3} dy = \int \frac{dx}{x^2 + 4}$$

gives:

$$\begin{aligned} \frac{1}{3}y &= \frac{1}{2}\tan^{-1}(x/2) + C \\ y &= \frac{3}{2}\tan^{-1}(x/2) + C \end{aligned}$$

(different C)

Now we use the initial condition:

$$\begin{aligned} y(2) &= \frac{3}{2}\tan^{-1}(2/2) + C \stackrel{set}{=} 0 \\ \Rightarrow \quad \frac{3}{2}\frac{\pi}{4} + C &= 0 \quad \Rightarrow \quad C = -\frac{3\pi}{8}. \end{aligned}$$

So, the final solution is

$$y = \frac{3}{2}\tan^{-1}(x/2) - \frac{3\pi}{8}.$$