APPM 4/5520

Solutions for Review Problems for Exam II, Sections 7.3-7.4

6. Let $x = \sec\theta$. Then $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$ and $dx = \sec\theta\tan\theta\, d\theta$. The integral becomes

$$\int \frac{\sec\theta\tan\theta}{\tan^4\theta}\, d\theta = \int \frac{\sec\theta}{\tan^3\theta}\, d\theta$$

Yuch! Could we play around with trig identities until we make this work? Yup. But... we could try partial fractions too– don't forget about this option!

$$\frac{1}{[(x+1)(x-1)]^2} = \frac{1}{(x+1)^2(x-1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

The solution is $A = 1/4$, $B = 1/4$, $C = -1/4$, and $D = 1/4$. So the integral becomes

$$\frac{1}{4}\left[\int \frac{1}{x+1}\, dx + \int \frac{1}{(x+1)^2}\, dx - \int \frac{1}{x-1}\, dx + \int \frac{1}{(x-1)^2}\, dx\right]$$

Now each one is a simple u-substitution. The answer is

$$\int \frac{dx}{(x^2-1)^2} = \frac{1}{4}\left[\ln|x+1| - \frac{1}{x+1} - \ln|x-1| - \frac{1}{x-1}\right] + C$$

7. Let $u = \cos\theta$. Then $du = -\sin\theta\, d\theta$ and the integral becomes

$$-\int \frac{du}{u^2 + u - 2}$$

Now it's a partial fractions problem. The answer is

$$\frac{1}{3}\ln\left|\frac{u+2}{u-1}\right| + C = \frac{1}{3}\ln\left|\frac{\cos\theta + 2}{\cos\theta - 1}\right| + C$$

8. Let $u = 2\sqrt{x}$. Then $du = 1/\sqrt{x}$ and the integral becomes

$$\int_{1/\sqrt{3}}^{1} \frac{2\, du}{1+u^2}$$

$$= 2\tan^{-1} u\Big|_{1/\sqrt{3}}^{1} = 2\left(\frac{\pi}{6} - \frac{\pi}{4}\right) = \frac{\pi}{6}$$

9. Let $t = 2\tan\theta$ for $-\pi/2 < \theta, \pi/2$. Then $dt = 2\sec^2\theta\, d\theta$ and $t^2 + 4 = 4\tan^2\theta + 4 = 4\sec^2\theta$. The integral becomes

$$\int \frac{8\tan^3\theta \cdot 2\sec^2\theta\, d\theta}{2\sec\theta} = 8\int \tan^3\theta\sec\theta\, d\theta$$

$$= 8\int \frac{\sin^3\,\theta}{\cos^4\,\theta}\,d\theta = 8\int \frac{\sin^{23}\,\theta\,\sin\,\theta}{\cos^4\,\theta}\,d\theta = 8\int \frac{(1-\cos^2\,\theta)\cdot\sin\,\theta}{\cos^4\,\theta}\,d\theta$$

Let $u = \cos\,\theta$. Then $du = -\sin\,\theta\,d\theta$ and the integral becomes

$$-8\int \frac{1-u^2}{u^4}\,du = 8\int \frac{u^2-1}{u^4}\,du = 8\int \left[\frac{1}{u^2}-\frac{1}{u^4}\right]\,du$$

$$= 8\left[-\frac{1}{u}+\frac{1}{3u^3}\right]+C = 8\left[-\frac{1}{u}+\frac{1}{3u^3}\right]+C = 8\left[-\frac{1}{\cos\,\theta}+\frac{1}{3\cos^3\,\theta}\right]+C$$

$$= 8\left[-\sec\,\theta+\frac{1}{3}\sec^3\,\theta\right]+C = 8\left[-\frac{\sqrt{t^2+4}}{2}+\frac{1}{3}\frac{(t^2+4)^{3/2}}{8}\right]+C$$

10. Let $x = \frac{1}{3}\sin\,\theta$ for $-\pi/2 \le \theta \le \pi/2$. Then $dx = \frac{1}{3}\cos\,\theta$ and $1-9x^2 = \cos^2\,\theta$. The integral becomes

$$\int \sqrt{\cos^2\,\theta}\cdot\frac{1}{3}\cos\,\theta\,d\theta = \int \cos\,\theta\cdot\frac{1}{3}\cos\,\theta$$

since $-\pi/2 \le \theta \le \pi/2$.

$$\frac{1}{3}\int \cos^2\,\theta\,d\theta = \frac{1}{3}\int \frac{1 = \cos\,2\theta}{2}\,d\theta = \frac{1}{6}\int (1+\cos\,2\theta)\,d\theta$$

$$= \frac{1}{6}\left[\theta+\frac{1}{2}\sin\,2\theta\right]+C = \frac{1}{6}\left[\theta+\sin\,\theta\,\cos\,\theta\right]+C = \frac{1}{6}\left[\sin^{-1}(3x)+3x\cdot\sqrt{1-9x^2}\right]+C$$