

APPM 4/5520

Solutions to “Some Review Problems for Exam III”

1. False. Example: the harmonic series.
2. True by direct comparison:  $a_n(-2)^n \leq a_n6^n$
3. True by direct comparison.
4. False since  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ . (No info!)
5. True since  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$ .
6. False:  $\sum a_n$  is smaller than  $\sum b_n$  and so it may converge.
7. True:  $1/e = e^{-1}$  and the Maclaurin series for  $e^x$  is  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  which converges for all  $x$ .  
Therefore,  $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ .
8. False: A power series has the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

9. False: It is still necessary that  $R_n(x) \rightarrow 0$ .
10. True
11. ??? It should have read “If  $\sum a_n$  is divergent then  $\sum |a_n|$  is divergent.” Then, the answer is “True”. (The second sum is bigger.)
12. True because the coefficient of  $x^3$  is given by  $f'''(0)/3!$ . Or, perhaps more simply, you could compute  $f'''(0)$  by taking three derivatives of the given  $f$  and plugging in zero!
13. False. Example:  $a_n = n$  and  $b_n = -n$ .
14. There is no question 14!
15. Converges to  $1/2$ .
16. Converges to 5.
17. Diverges to  $\infty$ .
18. Diverges by L'Hopital's rule.
19. Diverges, limit does not exist.

20. Converges to zero by the “squeeze theorem”.

$$0 \leq \left| \frac{\sin n}{n} \right| \leq \frac{1}{n}$$

and  $1/n \rightarrow 0$  as  $n \rightarrow \infty$ . Or, don't use absolute values:

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

Both  $-1/n$  and  $1/n$  go to zero.

21. Rewrite as  $[(1 + 3/n)^n]^4$ . Converges to  $[e^3]^4 = e^{12}$ .

22. Converges: Rewrite as  $(-1)^n 10^n/n!$  and note that  $10^n/n!$  goes to zero:

$$\frac{10^n}{n!} = \frac{10 \cdot 10 \cdot 10 \cdots 10}{1 \cdot 2 \cdot 3 \cdots 10} \cdot \frac{10 \cdot 10 \cdots 10}{11 \cdot 12 \cdots n} \leq 10^{10} \left( \frac{10}{11} \right)^{n-10} \rightarrow 0$$

23. Diverges by limit comparison test with the harmonic series.

24. Converges: Limit comparison test with  $b_n = n^2/n^4 = 1/n^2$ .

25. Converges by alternating series test:  $a_n = (-1)^n b_n$  where  $b_n = 1/n^{1/4}$ . We have  $b_{n+1} \leq b_n$  and  $b_n \rightarrow 0$ .

26. Converges by ratio test.

27. Converges by root test.

28. Diverges by  $n$ -th term test for divergence.

29.

$$\left| \frac{\sin n}{1+n^2} \right| \leq \frac{1}{1+n^2} \leq \frac{1}{n^2}$$

Therefore

$$\sum \left| \frac{\sin n}{1+n^2} \right| \leq \sum \frac{1}{n^2} < \infty$$

( $p$ -series with  $p = 2$ ) Therefore the original series is absolutely convergent and, hence, convergent.

30. Converges by integral test.

31. Converges by ratio test.

32. Converges by alternating series test.  $a_n = (-1)^{n+1} b_n$  where  $b_n = \ln n/\sqrt{n}$ . Can show that  $b_n$  is eventually decreasing (so  $b_{n+1} \leq b_n$ ) by considering the function  $f(x) = \ln x/\sqrt{x}$  and its derivative. Also, need to show that  $b_n \rightarrow 0$  using L'Hopital's rule.

33. Diverges by ratio test.

34. Rationalize by multiplying top and bottom by  $\sqrt{n+1} + \sqrt{n-1}$  to get

$$a_n = \frac{2}{n(\sqrt{n+1} + \sqrt{n-1})}.$$

Now do the limit comparison test with  $b_n = 1/n^{3/2}$ . Since  $\lim_{n \rightarrow \infty} a_n/b_n = 1$  and  $\sum b_n$  is a convergent  $p$ -series, the original series converges.