SIAM J. Numer. Anal.
Vol. 50, No. 4, pp. 1849–1860

TWO RESULTS CONCERNING THE STABILITY OF STAGGERED MULTISTEP METHODS*

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Abstract. Staggered spatial node layouts are widely used in the context of solving wave equations. When this idea of staggering is extended to time integrators for wave equations, one finds methods that offer both better accuracy and less restrictive stability conditions for an equivalent computational cost. Here, we prove two results concerning staggered multistep methods. We first prove that staggered backwards differentiation methods are unstable for all orders $p \geq 5$. We then extend Dahlquist's first stability barrier to explicit staggered multistep methods.

Key words. staggered finite difference methods, time integrators, linear multistep methods, staggered backwards differentiation method, Dahlquist's first stability barrier

AMS subject classifications. 65L06, 65L12, 65L20, 65M06, 65M12

DOI. 10.1137/110837541

1. Introduction.

It is well known that for approximating spatial derivatives, approximations for odd-order derivatives are generally more accurate at locations halfway between grid points than at grid points, with the opposite true for even-order derivatives [4], [7]. In [9], we developed and analyzed ODE solvers (in particular, multistep and Runge–Kutta methods) that utilized this same idea. For a given order of accuracy, staggered multistep methods have about ten times less local truncation error and stability domains that extend approximately 2–8 times as far on the imaginary axis when compared to their nonstaggered counterparts [9]. This improved accuracy and stability with no additional computational or storage cost makes such methods ideal when they can be applied.

Staggered methods are primarily of use in approximating solutions to linear wave equations since they can be formulated with a grid which is staggered in space and/or time. In general, staggered methods can be applied to equations which can be written in the form

$$\vec{U}\,'(t) = \vec{F}\left(t, \vec{V}(t)\right), \qquad \vec{V}\,'(t) = \vec{G}\left(t, \vec{U}(t)\right) \tag{1.1}$$

for some choice of $\vec{U}, \vec{V}, \vec{F}(t, \vec{V})$, and $\vec{G}(t, \vec{V})$.

We give two examples; other linear wave equations, including problems in three dimensions, can be treated similarly. Figure 1.1 shows four different ways to lay out the grid of unknowns u and v for the one-dimensional acoustic wave equation

*Received by the editors June 15, 2011; accepted for publication (in revised form) May 14, 2012; published electronically July 3, 2012.
http://www.siam.org/journals/sinum/50-4/83754.html

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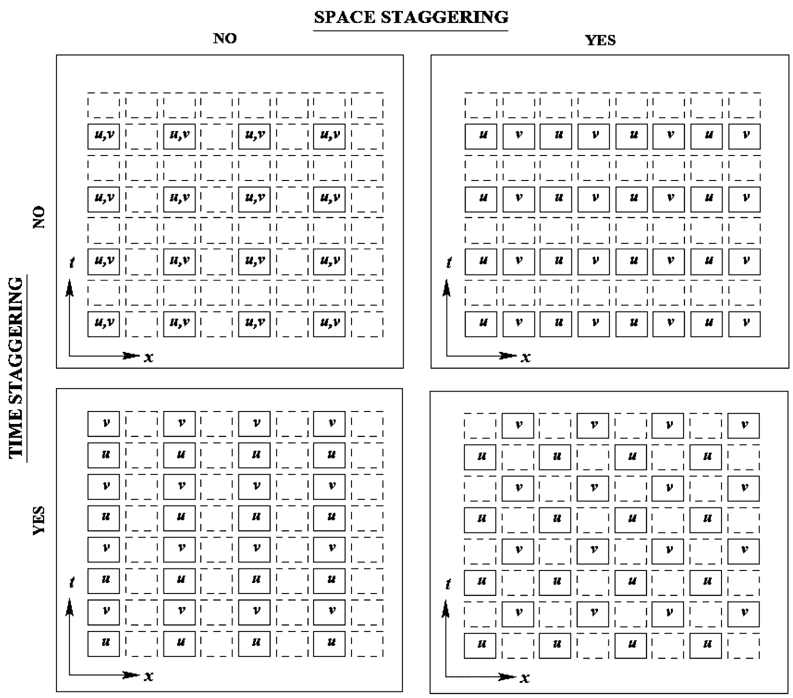


FIG. 1.1. *Representative samples of various spatial/time grid layouts for the one-dimensional wave equation* (1.2). *Each box represents a half-integer spatial or time increment.*

$$\begin{aligned} \frac{\partial u}{\partial t} &= c\frac{\partial v}{\partial x}, \\ \frac{\partial v}{\partial t} &= c\frac{\partial u}{\partial x}. \end{aligned} \tag{1.2}$$

One can choose to utilize time staggering, space staggering, both, or neither. In each case, the space-time density of data is exactly the same. If one wants to incorporate staggering in time, the variables u and v must exist on interlaced time intervals (e.g., u exists on integer time levels, while v exists on half-integer time levels.)

One can also apply staggering to the two-dimensional elastic wave equation

$$\begin{aligned} \rho\frac{\partial u}{\partial t} &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}, \\ \rho\frac{\partial v}{\partial t} &= \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y}, \\ \frac{\partial f}{\partial t} &= (\lambda + 2\mu)\,\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y}, \\ \frac{\partial g}{\partial t} &= \mu\frac{\partial v}{\partial x} + \mu\frac{\partial u}{\partial y}, \\ \frac{\partial h}{\partial t} &= \lambda\frac{\partial u}{\partial x} + (\lambda + 2\mu)\,\frac{\partial v}{\partial y}. \end{aligned}$$

To stagger this equation in time, we must again split the variables into two groups that exist on interlaced time levels (e.g., u and v on integer time levels and f, g, and h on half-integer time levels). It is also possible to stagger this equation in just space

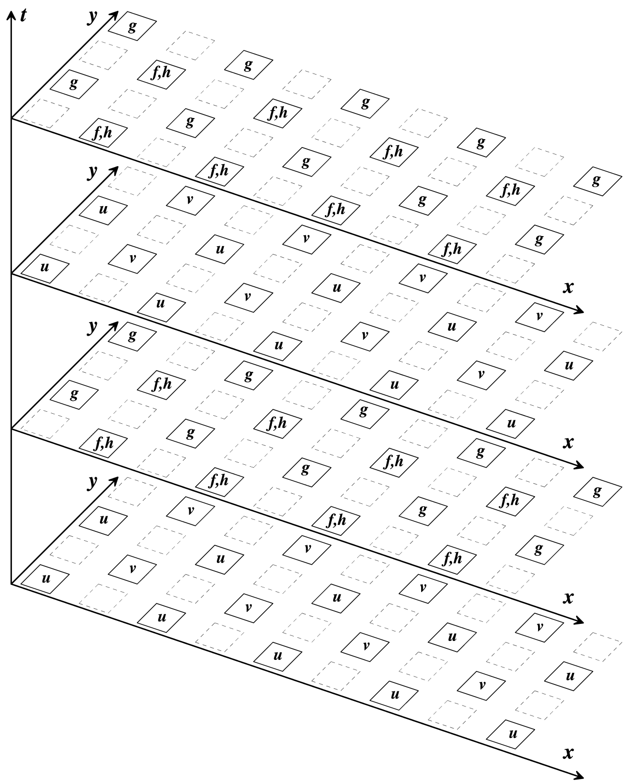


FIG. 1.2. *Representative sample of a spatial-staggered, time-staggered grid for the two-dimensional elastic wave equation* (1.3). *Each box/level represents a half-integer spatial/time increment.*

or in both time and space; an illustration of the latter is given in Figure 1.2. For more information on using staggering in space and/or time, see, for example, [5], [6], [7], [8], [9]. In particular, the first four of these references address staggering for the three-dimensional Maxwell equations.

One classical result for (nonstaggered) multistep methods is that backwards differentiation (BDF) methods are unstable for all orders $p \geq 8$ [1], [2]. In section 2 of this paper, we extend the proofs of [10] and [11] to establish that staggered backwards differentiation (BDS) methods are unstable for orders $p \geq 5$.

Another classical result is Dahlquist's first stability barrier [3], [15], [16], which established that if a linear m-step method is to be zero-stable, then its order of accuracy p must satisfy

$$p \leq \begin{cases} m, & \text{explicit method,} \\ m+1, & m \text{ odd, implicit method,} \\ m+2, & m \text{ even, implicit method.} \end{cases} \tag{1.3}$$

This result was subsequently reproven using order stars in [13]. In section 3, we extend the proofs of [3] and [15] to find the corresponding barrier for explicit staggered methods.

2. Stability of staggered backwards differentiation methods. A standard (nonstaggered) implicit n-step backwards differentiation method for approximating solutions to $\frac{dy}{dt} = f(t,y)$ has the form $\sum_{k=1}^{n} \frac{1}{k}\nabla^k y_k = hf_n$, where $\nabla y_k = y_k - y_{k-1}$ and $f_n = f(t_n)$. This method has order n; its associated generating polynomials are given by (see, e.g., p. 27 of [12]) $\sigma(\xi) = \xi^n$ and

$$\rho(\xi) = \sum_{k=1}^{n} \frac{1}{k}\xi^{n-k}(\xi-1)^k. \tag{2.1}$$

An n-step BDS method has the form

$$\sum_{k=0}^{n} A_k y_k = hf_{n-1/2}, \tag{2.2}$$

where the coefficients A_k are chosen to make the method of maximal order $p = n$ (except for $n = 1$, leapfrog, which has order $p = 2$). These staggered methods are explicit as one uses (known) derivative information halfway between the grid points to advance the solution, as illustrated in section 1. In Table 2.1, we show all stable BDS methods.

A finite difference method is called zero-stable if the roots of the generating polynomial $\rho(\xi)$ satisfy the following:

1. All roots ξ_i satisfy $|\xi_i| \le 1$, and
2. all roots ξ_i with $|\xi_i| = 1$ are simple roots.

If either of these two conditions is violated, the method is unstable; this is equivalent to $\rho(\xi)$ having either a root ξ_i outside the unit circle or a multiple root on the unit circle.

THEOREM 2.1. *BDS methods are zero-stable for orders* $2 \le p \le 4$ *and are unstable for orders* $p \ge 5$.

Proof. From (2.2), we see that $\sigma(\xi) = \xi^{n-1/2}$ for a general n-step BDS method. We seek to find $\rho(\xi)$ for this method. Theorem 2.1 of Iserles [12] says that for a multistep method to be of order $p \ge 1$, then as $\xi \to 1$, we must have

$$\rho(\xi) - \sigma(\xi)\ln\xi = \mathcal{O}\left(|\xi-1|^{p+1}\right). \tag{2.3}$$

Substituting $\xi = 1/v$ and $\sigma(\xi) = \xi^{n-1/2}$, letting $p = n$ (as we want this method to be of maximal order n), and multiplying through by v^n gives

$$v^n\rho\left(\frac{1}{v}\right) + v^{1/2}\ln v = \mathcal{O}\left(\frac{1}{v}|v-1|^{n+1}\right) = \mathcal{O}\left(|v-1|^{n+1}\right) \tag{2.4}$$

as $v \to 1$. We also have

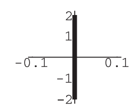
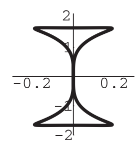
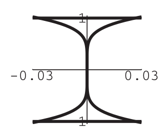
$$\ln v = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}(v-1)^m$$

and

$$v^{1/2} = \sum_{j=0}^{\infty} \binom{1/2}{j}(v-1)^j$$

TABLE 2.1

Staggered backwards differentiation (BDS) time integrators. The normalized local truncation error for BDSp is $C_p h^{p+1} f^{(p+1)}(\eta)$, where C_p is the error constant. This table shows all zero-stable BDS methods. For the stencils, an open square represents an unknown function value, while a closed square/circle stands for a known function/derivative value. ISB is the imaginary stability boundary, which gives the maximum extent of the stability domain along the imaginary axis. Note: this table previously appeared as Table 4.1 in [9].

Name	Stencil	Coefficients	Error constant	Stability domain	ISB
BDS2 (leapfrog)	□ ● ■	$\left[\begin{array}{c\|c} 1 & 1 \\ -1 & \end{array}\right]$	$\frac{1}{24}$		2
BDS3	□ ● ■ ■ ■	$\left[\begin{array}{c\|c} 1 & \frac{24}{23} \\ -\frac{21}{23} & \\ -\frac{3}{23} & \\ \frac{1}{23} & \end{array}\right]$	$\frac{1}{24}$		$\frac{5}{3} \simeq 1.667$
BDS4	□ ● ■ ■ ■ ■	$\left[\begin{array}{c\|c} 1 & \frac{12}{11} \\ -\frac{17}{22} & \\ -\frac{9}{22} & \\ \frac{5}{22} & \\ -\frac{1}{22} & \end{array}\right]$	$\frac{71}{1920}$		1

from series expansion. Then we find that $v^{1/2} \ln v = \sum_{k=0}^{\infty} C_k (v-1)^n$ from series multiplication, where $C_0 = 0$ and, for $k \geq 1$,

$$C_k = \sum_{i=0}^{k-1} \binom{1/2}{i} \frac{(-1)^{k-i-1}}{k-i}.$$

From (2.4), we find that $v^n \rho(\frac{1}{v}) = -\sum_{k=0}^{n} C_k (v-1)^k$ since $\rho(\xi)$ consists of $n+1$ terms. Using $C_0 = 0$ and $v = \frac{1}{\xi}$ gives

$$\rho(\xi) = \sum_{k=1}^{n} (-1)^{k+1} C_k \xi^{n-k} (\xi - 1)^k \equiv \sum_{k=1}^{n} B_k \xi^{n-k} (\xi - 1)^k, \tag{2.5}$$

where B_k is defined by

$$B_k = \sum_{i=0}^{k-1} \binom{1/2}{i} \frac{(-1)^i}{k-i}. \tag{2.6}$$

Note that for nonstaggered BDF methods, $B_k = \frac{1}{k}$ from (2.1), a significantly simpler expression.

Next, as in [10] and [11], make the substitution $\xi = \frac{1}{1-z}$, which transforms the unit circle $\xi = 1$ to the circle $|z-1| = 1$ and reverses the inside and outside. For a given order n, let

$$P_n(z) = \frac{(1-z)^n}{z}\rho\left(\frac{1}{1-z}\right) = \sum_{k=1}^{n} B_k z^{k-1}, \tag{2.7}$$

where we have used (2.5) and B_k is defined in (2.6).

Each BDS method of order n now has an order $n-1$ polynomial $P_n(z)$ associated with it. The zero-stability conditions imply that in order to establish that BDS methods are zero-stable for $n \le 4$, we must show that the roots of $P_n(z)$ are outside $|z-1| = 1$ for $n \le 4$. To show that BDS methods are unstable for $n \ge 5$, we must show that $P_n(z)$ has at least one root inside $|z-1| = 1$ (or a double root on its boundary) for $n \ge 5$.

Let us examine some of these polynomials which are defined by (2.7) and (2.6):

$$\begin{aligned} P_1(z) &= P_2(z) = 1, \\ P_3(z) &= 1 - \frac{1}{24}z^2, \\ P_4(z) &= 1 - \frac{1}{24}z^2 - \frac{1}{24}z^3, \\ P_5(z) &= 1 - \frac{1}{24}z^2 - \frac{1}{24}z^3 - \frac{71}{1920}z^4, \\ P_6(z) &= 1 - \frac{1}{24}z^2 - \frac{1}{24}z^3 - \frac{71}{1920}z^4 - \frac{31}{960}z^5. \end{aligned} \tag{2.8}$$

From these polynomials, we observe that $P_1(z)$ and $P_2(z)$ trivially satisfy the condition for zero-stability. We also notice that the coefficients B_k depend only on k and not on the order of the method n, which can also be seen in (2.6).

In Figure 2.1, we show the $n-1$ complex roots of $P_n(z)$ for $3 \le n \le 8$. We observe that for $n \le 4$, all roots are outside $|z-1| = 1$; this can be calculated directly from (2.8) to prove that BDS methods are stable for orders $2 \le n \le 4$. It appears that for each $n \ge 3$, $P_n(z)$ has exactly one positive real root which monotonically decreases towards $z = 1$; for $n \ge 5$, that real root is inside $|z-1| = 1$. In contrast, the polynomials for the nonstaggered BDF methods considered in [10] and [11] had a pair of complex roots which slowly moved inside $|z-1| = 1$ to cause the instability. Our proof now diverges from [10] and [11].

We formalize our observations, first showing that each polynomial $P_n(z)$ given by (2.7) and (2.6) has exactly one real positive root. From (2.6), one can compute directly that $B_1 = 1$ and $B_2 = 0$. For $k \ge 3$, we can rewrite (2.6) to find that

$$B_k = -\frac{1}{2^{2k-3}}\frac{(2k-3)!}{k!(k-2)!}\sum_{i=2}^{k-1}\frac{1}{2i-1} < 0. \tag{2.9}$$

From (2.7), we have $P_n(0) = B_1 = 1$. Since all other coefficients of $P_n(z)$ are negative, $P_n(z)$ must decrease monotonically to $-\infty$ as $z \to \infty$ for fixed $n \ge 3$. Thus, for $n \ge 3$, each $P_n(z)$ will have exactly one positive root which we call $\hat{z}_n$. By direct calculation from (2.8), $\hat{z}_3 = 2\sqrt{6} \approx 4.90$, $\hat{z}_4 \approx 2.59 > 2$, and $\hat{z}_5 \approx 1.95 < 2$.

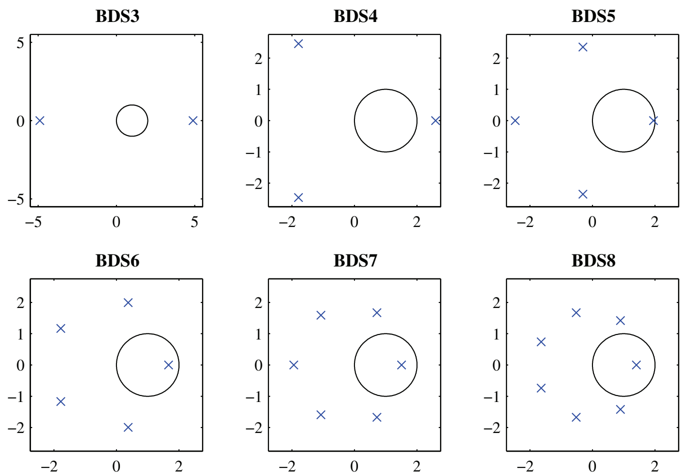


FIG. 2.1. *The $n-1$ roots of $P_n(z)$ (which is associated with BDSn) are shown in the complex plane for $3 \le n \le 8$ along with the circle $|z-1| = 1$. For $n \le 4$, all roots are outside the circle. For $n \ge 5$, one root is inside the circle, making these methods unstable; note that this root appears to be real and monotonically decreasing.*

Since $P_n(\hat{z}_n) = 0$, from (2.7) we have $P_{n+1}(\hat{z}_n) = B_{n+1}(\hat{z}_n)^n < 0$. Because $P_{n+1}(0) = B_1 = 1$ and $P_{n+1}(\hat{z}_n)$ have opposite signs, we find that the root of $P_{n+1}(z)$, $\hat{z}_{n+1}$, must be between 0 and $\hat{z}_n$. Therefore, for $n \ge 3$, $\hat{z}_n$ is a monotonically decreasing sequence which is bounded below by 0.

We thus find that BDS methods always have a (real) root inside $|z-1| = 1$ for $n \ge 5$. Therefore BDS methods are zero-stable for orders $n \le 4$ and unstable for orders $n \ge 5$. This completes the proof of Theorem 2.1. ∎

Note: The characteristic polynomials mentioned in this article are only needed for the analysis of the staggered ODE solvers. The user of these schemes need only be concerned with the resulting information, as summarized in Table 2.1 and, for other staggered schemes, in Tables 4.2 and 9.1 of [9], for example.

3. The staggered analogue of Dahlquist's first stability barrier. As in [9], we consider only explicit methods since no plausible such scheme has been proposed. For staggered explicit multistep methods, the following theorem is the analogue of Dahlquist's first stability barrier (1.3).

THEOREM 3.1. *The order p of an explicit stable staggered m-step method satisfies*

$$p \le \begin{cases} m, & m \text{ an even integer,} \\ m + \frac{1}{2}, & m \text{ a half-integer,} \\ m+1, & m \text{ an odd integer.} \end{cases}$$

For staggered methods, m is an integer when the oldest information utilized is a function value, as in BDS. When the oldest information used is instead a derivative value, as in staggered Adams–Bashforth methods [9], m is a half-integer. This theorem was previously given in [8] and [9], but the proof has only appeared in [8]. Our proof of this theorem follows the proofs of Dahlquist [3] and Jeltsch and Nevanlinna [14] which were done for nonstaggered methods.

We begin with several lemmas, the first of which parallels a lemma on p. 50 of [3].

LEMMA 3.2. *The expansion of*

$$\frac{z}{\sqrt{1-z^2}\ln\frac{1+z}{1-z}} = \sum_{j=0}^{\infty} d_j z^j \tag{3.1}$$

satisfies $d_{2j+1} = 0$ *and* $d_{2j} > 0$ *for* $j \geq 0$.

Proof. One can see that $d_0 = \frac{1}{2}$ by considering the limit of the left-hand side as $z \to 0$. Also, $d_{2j+1} = 0$ because the left-hand side of (3.1) is an even function.

We divide both sides of (3.1) by z and then substitute $z = \frac{1}{w}$, giving

$$\frac{w}{\sqrt{w^2-1}\,\ln\left(\frac{w+1}{w-1}\right)} = \frac{w}{2} + \sum_{k=0}^{\infty} d_{2k+2} w^{-(2k+1)}.$$

By Cauchy's integral formula,

$$d_{2k+2} = \frac{1}{2\pi i}\oint_C w^{2j} \frac{w}{\sqrt{w^2-1}\,\ln\left(\frac{w+1}{w-1}\right)} dw$$

for $k \geq 0$, where C is an arbitrary curve enclosing $(-1,1)$ on the real axis. By taking our branch cut on $(-1,1)$ of the real axis, we find that

$$\begin{aligned} d_{2k+2} &= \frac{1}{2\pi i}\left[\int_{-1}^{1} \frac{x^{2k+1}(i)}{\sqrt{1-x^2}\left(i\pi + \ln\left(\frac{1+x}{1-x}\right)\right)} dx - \int_{-1}^{1} \frac{x^{2k+1}(-i)}{\sqrt{1-x^2}\left(-i\pi + \ln\left(\frac{1+x}{1-x}\right)\right)}\right] dx \\ &= \frac{1}{\pi}\int_{-1}^{1} \frac{x^{2k+1}\ln\left(\frac{1+x}{1-x}\right)}{\sqrt{1-x^2}\left(\pi^2 + \ln^2\left(\frac{1+x}{1-x}\right)\right)} dx. \end{aligned}$$

Because this last integrand is nonnegative on $(-1,1)$, we find that $d_{2k+2} > 0$ for $k \geq 0$. Since $d_0 = \frac{1}{2} > 0$, we conclude that $d_{2j} > 0$ for $j \geq 0$, thus proving Lemma 3.2. ∎

COROLLARY 3.3. *In the expansion*

$$\frac{z\sqrt{\frac{1+z}{1-z}}}{\ln\left(\frac{1+z}{1-z}\right)} = \sum_{j=0}^{\infty} \gamma_j z^j, \tag{3.2}$$

we have $\gamma_j > 0$ *for* $j \geq 0$.

Proof. We note that

$$\frac{z\sqrt{\frac{1+z}{1-z}}}{\ln\left(\frac{1+z}{1-z}\right)} = (1+z)\frac{z}{\sqrt{1-z^2}\ln\left(\frac{1+z}{1-z}\right)} = (1+z)\sum_{j=0}^{\infty} d_{2j} z^{2j},$$

where d_{2j} was defined in (3.1). Then, by Lemma 3.2, $\gamma_{2j} = \gamma_{2j+1} = d_{2j} > 0$. ∎

LEMMA 3.4 (Dahlquist [3]). *If* $\rho(z) \equiv \sum_{j=0}^{m} a_j z^j$ *for a stable multistep method, then all coefficients* a_j *which are nonzero have the same sign.*

The proof of this lemma can be found in [3].

We now proceed with the proof of Theorem 3.1.

3.1. Proof of Theorem 3.1.

3.1.1. Case 1: m is a half-integer. For this case, we can represent the generating polynomials as

$$\rho(\xi) = \xi^{1/2}\left[\alpha_0 + \alpha_1\xi + \cdots + \alpha_{m-1/2}\xi^{m-1/2}\right],$$
$$\sigma(\xi) = \left[\beta_0 + \beta_1\xi + \cdots + \beta_{m-1/2}\xi^{m-1/2}\right].$$

We make the bilinear transformation $\xi = \frac{1+z}{1-z}$ and define the functions

$$\begin{aligned} r(z) &\equiv \left(\frac{1-z}{2}\right)^{m-1/2}\rho\left(\frac{1+z}{1-z}\right) \\ &= \frac{1}{2^{m-1/2}}\sqrt{\frac{1+z}{1-z}}\left[\alpha_0(1-z)^{m-1/2} + \alpha_1(1-z)^{m-3/2}(1+z)\right. \\ &\qquad\qquad \left.+\cdots+\alpha_{m-1/2}(1+z)^{m-1/2}\right] \\ &\equiv \sqrt{\frac{1+z}{1-z}}\sum_{i=0}^{m-1/2} a_i z^i \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} s(z) &\equiv \left(\frac{1-z}{2}\right)^{m-1/2}\sigma\left(\frac{1+z}{1-z}\right) \\ &= \frac{1}{2^{m-1/2}}\left[\beta_0(1-z)^{m-1/2} + \beta_1(1-z)^{m-3/2}(1+z)\right. \\ &\qquad\qquad \left.+\cdots+\beta_{m-1/2}(1+z)^{m-1/2}\right] \\ &\equiv \sum_{i=0}^{m-1/2} b_i z^i. \end{aligned} \tag{3.4}$$

We note that because $\rho(\xi = 1) = 0$ (from consistency), we have $r(z = 0) = 0$. Thus $a_0 = 0$.

Because we want a stable method, all roots ξ of $\rho(\xi)$ must be inside the unit disk, with all roots on the unit circle simple. So, all roots z of $r(z)$ must lie in the closed left-hand plane, with all roots on the imaginary axis simple. Then, by Lemma 3.4, all coefficients a_i which are nonzero must have the same sign. Since $a_1 = r'(z = 0) = 2^{3/2-m} > 0$, we have that $a_j \geq 0$ for all $j \geq 1$.

The condition for a multistep method to be of order p (2.3) can be written as

$$\frac{\rho(\xi)}{\ln\xi} - \sigma(\xi) = c_{p+1}(\xi-1)^p + \mathcal{O}\left[(\xi-1)^{p+1}\right] \tag{3.5}$$

for some $c_{p+1} \neq 0$. This can be rewritten in terms of z, $r(z)$, and $s(z)$ as

$$\frac{r(z)}{z\sqrt{\frac{1+z}{1-z}}}\left[\frac{z\sqrt{\frac{1+z}{1-z}}}{\ln\left(\frac{1+z}{1-z}\right)}\right] - s(z) = 2^{p-m+1/2}c_{p+1}z^p + \mathcal{O}(z^{p+1}).$$

Using the definitions of a_i, b_i, and γ_j given in (3.3), (3.4), and (3.2), this becomes

$$\left(\sum_{i=1}^{m-1/2} a_i z^{i-1}\right)\left(\sum_{j=0}^{\infty} \gamma_j z^j\right) - \sum_{i=0}^{m-1/2} b_i z^i = 2^{p-m+1/2} c_{p+1} z^p + O(z^{p+1}). \tag{3.6}$$

The first term in the expansion of the product of series in (3.6) that cannot possibly be cancelled by a term in the series for $s(z)$ is the $z^{m+1/2}$ term. We let $p = m + 1/2$ and consider the coefficients of the $z^{m+1/2}$ term on both sides of the equation to find that

$$\sum_{j=1}^{m-1/2} a_j \gamma_{m+3/2-j} = 2c_{m+3/2}.$$

Then, because $\gamma_j > 0$ from Corollary 3.3 and $a_j \geq 0$ for all $j > 0$ from Lemma 3.4, we have

$$c_{m+3/2} = \frac{1}{2} \sum_{j=1}^{m-1/2} a_j \gamma_{m+3/2-j} \geq \frac{1}{2} a_1 \gamma_{m+1/2} > 0.$$

Since $c_{m+3/2} \neq 0$, we find that p cannot equal (or be larger than) $m + \frac{3}{2}$. Thus, the order p of an m-step method must satisfy $p \leq m + \frac{1}{2}$ when m is a half-integer.

3.1.2. Case 2: m is an integer. When m is an integer, we can represent our generating polynomials as

$$\rho(\xi) = [\hat{\alpha}_0 + \hat{\alpha}_1 \xi + \cdots + \hat{\alpha}_m \xi^m],$$
$$\sigma(\xi) = \xi^{1/2} \left[\hat{\beta}_0 + \hat{\beta}_1 \xi + \cdots + \hat{\beta}_{m-1} \xi^{m-1}\right].$$

After making the transformation $\xi = \frac{1+z}{1-z}$, we define the functions

$$\begin{aligned} \hat{r}(z) &\equiv \left(\frac{1-z}{2}\right)^m \rho\left(\frac{1+z}{1-z}\right) \\ &= \frac{1}{2^m}\left[\hat{\alpha}_0(1-z)^m + \hat{\alpha}_1(1-z)^{m-1}(1+z) + \cdots + \hat{\alpha}_m(1+z)^m\right] \\ &\equiv \sum_{i=1}^{m} \hat{a}_i z^i \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \hat{s}(z) &\equiv \left(\frac{1-z}{2}\right)^m \sigma\left(\frac{1+z}{1-z}\right) \\ &= \frac{1}{2^m}\sqrt{1-z^2}\left[\hat{\beta}_0(1-z)^m + \hat{\beta}_1(1-z)^{m-1}(1+z) + \cdots + \hat{\beta}_{m-1}(1+z)^{m-1}\right] \\ &\equiv \sqrt{1-z^2} \sum_{i=0}^{m-1} \hat{b}_i z^i. \end{aligned} \tag{3.8}$$

Again, we have $\hat{a}_0 = 0$ from consistency and $\hat{a}_j \geq 0$ for $j \geq 1$ from stability (noting that $\hat{a}_1 = 2^{1-m} > 0$) by Lemma 3.4. The order condition (3.5) becomes

$$\frac{\hat{r}(z)}{z}\left[\frac{z}{\sqrt{1-z^2}} \ln\left(\frac{1+z}{1-z}\right)\right] - \frac{\hat{s}(z)}{\sqrt{1-z^2}} = 2^{p-m} \hat{c}_{p+1} z^p + \mathcal{O}(z^{p+1}),$$

which can be rewritten as

$$\left(\sum_{i=1}^{m}\hat{a}_i z^{i-1}\right)\left(\sum_{j=0}^{\infty} d_j z^j\right) - \sum_{i=0}^{m-1}\hat{b}_i z^i = 2^{p-m}\hat{c}_{p+1}z^p + \mathcal{O}(z^{p+1}), \tag{3.9}$$

where $\hat{a}_i$, $\hat{b}_i$, and d_j are defined by (3.7), (3.8), and (3.1).

We know from Lemma 3.2 that $d_{2j} > 0$ and $d_{2j+1} = 0$. We consider the cases m even and m odd separately.

If m is even, then $d_m > 0$. The first term in the product of the two series in (3.9) that cannot be cancelled by a term in the series for $s(z)$ is the z^m term. We let $p = m$ and consider the coefficients of the z^m terms, giving

$$\hat{c}_{m+1} = \sum_{i=1}^{m}\hat{a}_i d_{m-i+1} \geq \hat{a}_1 d_m > 0.$$

Thus, $\hat{c}_{m+1} \neq 0$ and we find that such a method cannot have order $m+1$ (or higher). Thus, we find that $p \leq m$ for m even.

If m is odd, then $d_m = 0$ but $d_{m+1} > 0$. The first term in the product of the two series in (3.9) that cannot be cancelled by a term in the series for $s(z)$ is the z^{m+1} term. We then let $p = m+1$ and consider the coefficients of the z^{m+1} terms, giving

$$\hat{c}_{m+2} = \frac{1}{2}\sum_{i=1}^{m}\hat{a}_i d_{m-i+2} \geq \frac{1}{2}\hat{a}_1 d_{m+1} > 0.$$

Thus, $\hat{c}_{m+2} \neq 0$ and we find that such a method cannot have order $m+2$ (or higher). Thus, we find that $p \leq m+1$ for m odd.

This completes the proof of Theorem 3.1. ∎

3.2. Discussion. We note that staggered Adams–Bashforth methods, which are stable for all orders, have order $p = m + \frac{1}{2}$ and thus achieve equality for the case where m is a half-integer. From section 2, we know that BDS methods are stable for orders $p \leq 4$; they have $p = m$ and thus achieve equality for even integers $m < 5$. For $m = 1$, equality is achieved by the leapfrog method, which has order 2 (see Table 2.1). An example of a method which achieves equality for $m = 3$ is given by

$$\begin{aligned} y_{n+1} + (-27+24u)y_n - (-27+24u)y_{n-1} - y_{n-2} \\ = h\left(uf_{n+1/2} + (-24+22u)f_{n-1/2} + uf_{n-3/2}\right). \end{aligned}$$

This fourth order scheme with parameter u is stable for $u \in (1, 7/6)$ with stability domains consisting of only the origin. It is currently an open question whether there exist stable staggered methods which achieve equality for $m \geq 5$ or a method with useful stability domain for $m = 3$.

4. Conclusions. Staggered multistep methods can be highly effective for approximating solutions to linear wave equations. In this paper, we have proven two results concerning staggered multistep methods. First, staggered backwards differentiation methods are stable for orders $2 \leq p \leq 4$ and unstable for all orders $p \geq 5$. Second, we find that the order p of an explicit stable staggered m-step method satisfies

$$p \leq \begin{cases} m, & m \text{ an even integer}, \\ m + \frac{1}{2}, & m \text{ a half-integer}, \\ m+1, & m \text{ an odd integer}, \end{cases}$$

which is the staggered analogue of Dahlquist's first stability barrier.

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