Matter–wave interference in Bose–Einstein condensates: A dispersive hydrodynamic perspective

M.A. Hoefer a,*, P. Engels b, J.J. Chang b

a National Institute of Standards and Technology, Boulder, CO 80305, USA
b Washington State University, Department of Physics and Astronomy, Pullman, WA, 99164, USA

1. Introduction

Rich, nonlinear, dispersive wave dynamics abound in the study of Bose–Einstein condensates (BECs). In particular, the interaction of initially separated BECs, the BEC merging problem, has yielded interesting experimental results, including interference patterns enabling atomic interferometry and illustrative of the fundamental wave nature of matter [1,2]. Coherent nonlinear wave structures such as dispersive shock waves [3,4], solitons, vortices, and vortex rings [4–8] have also been observed in the context of BEC merging experiments. Motivated by these intriguing experimental results, we study the BEC merging problem from a hydrodynamic perspective. An asymptotic analysis of the governing Gross–Pitaevskii (GP) equation in the small dispersion limit, experiments, and three-dimensional (3D) simulations are used to demonstrate the dispersive hydrodynamic origins of matter–wave interference.

The merging problem we consider here has been studied theoretically in various contexts. The interference of BECs was proposed and studied in the linear regime in [9]. Numerical studies in 1D showed the existence of interference fringes when two BECs, initially residing in a double well potential, interact [10,11]. In [12], 1D simulations of the GP equation with two initially trapped, separated condensates were performed. It was argued that solitons are generated during the merging interaction that ensues. Indeed, in this work, we present 3D numerical as well as experimental data observing such soliton formation. Furthermore, the approximate, asymptotic (small dispersion) solution we calculate in this work demonstrates how these solitons naturally arise from a linear interference pattern when the BEC density is increased. The solution is a modulated elliptic function that reduces to linear, trigonometric waves in the small density limit. Thus, our results bridge the linear theory of matter–wave interference for sufficiently small densities [9–11] to a nonlinear theory that describes the interference pattern as a modulated train of solitons. Soliton trains in BECs have been studied previously in 1D with the aid of the Inverse Scattering Transform (IST) [13,14]. The IST was also used to gain information about the long time behavior and interference fringe spacing of two interacting Gaussian wave packets [15].

In this work, we consider, both theoretically and experimentally, the case of an elongated, anisotropically confined BEC, often referred to as cigar shaped, following the experiments of [4].
configuration is studied analytically in the quasi-one-dimensional (1D) regime. A dispersive hydrodynamic viewpoint is useful in this regard. Using Whitham averaging theory [16, 17] for the small-dispersion limit of the Nonlinear Schrödinger (NLS) equation, we asymptotically solve the BEC merging problem for piecewise-constant initial data that correspond to the merging interaction of two semi-infinite BECs. We demonstrate the emergence of a soliton train (interference pattern) due to the interaction of two rarefaction waves propagating through a region of zero density (vacuum). Via multiple scale arguments, this asymptotic result is extended to the 3D GP equation. New experimental results are presented and compared favorably with our asymptotic calculations and 3D numerical simulations of the GP equation.

The results we present here are closely related to dispersive shock waves (DSWs). DSWs evolve after a dispersive hydrodynamic system develops large gradients and "breaks". They are slowly varying, expanding, oscillatory wave solutions and were studied in the context of the small dispersion limit of the NLS equation in Refs. [18, 19] with the use of Whitham averaging theory. These and classical results of viscous shock wave theory were applied to study BEC DSWs in Refs. [20, 21]. Further studies of BEC DSWs were performed in [22, 23], with the latter including experimental observation of DSW structures. In Ref. [24], Whitham averaging theory was used to describe the interactions of DSWs. The merging problem considered here (two semi-infinite, steady BECs separated by the vacuum) is related to the so-called collision problem of Ref. [24] where two DSWs counter-propagate on a non-zero background density toward one another and interact leading to modulated quasi-periodic or 2-phase behavior. In contrast, the merging problem involves the interaction of rarefaction waves propagating on the vacuum leading to a modulated periodic or 1-phase region, the BEC interference pattern.

The organization of this article is as follows. Section 2 is concerned with the calculation of the asymptotic solution to the 1D merging problem for the case of piecewise-constant initial data. We then show how this result can be used to study the merging problem for a 3D BEC in strong anisotropic confinement in Section 3. Finally, we present new experimental results and 3D numerical simulations that qualitatively agree with the aforementioned asymptotic results in Section 4.

2. The merging problem: NLS equation

In this section, we present the asymptotic solution to the merging problem for the NLS equation. We introduce the so-called 0- and 1-phase representations of the solution. Following this, the initial-value problem corresponding to the merging of two semi-infinite BECs. We demonstrate the emergence of a soliton train (interference pattern) due to the interaction of two rarefaction waves propagating through a region of zero density (vacuum). Via multiple scale arguments, this asymptotic result is extended to the 3D GP equation. New experimental results are presented and compared favorably with our asymptotic calculations and 3D numerical simulations of the GP equation.

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The model equation we study in this section is the scaled 1D NLS equation,

\[ i\psi_t - \frac{\varepsilon^2}{2} \psi_{xx} + |\psi|^2 \psi = 0, \tag{1} \]

where \(0 < \varepsilon \ll 1\) is the small-dispersion parameter. The 1D NLS equation is a reduced model of BEC dynamics which is governed more generally by the GP equation [25–27]. The NLS equation can also be written in a form analogous to the shallow-water equations. The transformation

\[ \psi(x, t) = \sqrt{\rho(x, t)} \exp \left( i \frac{1}{\varepsilon} \int_0^x u(x', t) \, dx' \right), \]

along with the first two local conservation laws for the NLS equation, give [28],

\[ \rho_t + (\rho u_x) = 0, \]
\[ (\rho u_x) + \left( \frac{\rho u^2}{2} + \frac{1}{2} \rho^2 \right) = \frac{\varepsilon^2}{4} \rho (\log \rho)_{xx}, \tag{2} \]

which are exact. These dispersive hydrodynamic equations form the basis of our perspective and analysis of the merging problem. Indeed, if \( \varepsilon = 0 \), these are the well-known shallow water equations [29, 30]. The term multiplying \( \varepsilon^2 \) is a dispersive term, and so we are interested in the small dispersion limit, also referred to in quantum mechanics as the semiclassical limit. Due to symmetry properties of Eqs. (2), if the initial data satisfy

\[ \rho(-x, 0) = \rho(x, 0), \quad u(-x, 0) = -u(x, 0), \tag{3} \]

then these symmetries are preserved for all time.

2.1. 0- and 1-phase asymptotic solutions

We will be interested in oscillatory solutions due to the interference of two separated BECs; therefore, Whitham modulation theory will be useful. We consider two types of asymptotic solutions to Eq. (1). These are exact traveling-wave solutions with one or two phase variables. The Whitham modulation equations result from assuming that the 0- or 1-phase wave’s parameters vary slowly in space and time with respect to the phase variables.

A detailed description of the modulated 0- and 1-phase solutions to the NLS equation in the framework of Whitham theory has been given, e.g., in Refs. [17–19, 23]. We briefly summarize this below. The 0-phase solution is written

\[ \Psi_0(x, t) = \sqrt{\rho(x, t)} e^{i\omega_0}, \quad \rho = \left| \omega_0 - \frac{1}{2} k_0 \right|^2, \quad u = k_0, \]

\[ \frac{\partial \theta_0}{\partial x} = k_0/\varepsilon, \quad \frac{\partial \theta_0}{\partial t} = -\omega_0/\varepsilon, \]

\[ \omega_0 = \omega_0(x, t), \quad k_0 = k_0(x, t), \tag{4} \]

because the only fast variable is the trivial, exponential phase \( \theta_0 \), \( x \) and \( t \) are slow variables here and throughout this work. Eqs. (2) with \( \varepsilon = 0 \) are known as the shallow water equations with fluid height \( \rho \) and local fluid velocity \( u \). The shallow water equations may be conveniently written in Riemann invariant form,

\[ \frac{\partial r_+}{\partial t} + v_+ \frac{\partial r_+}{\partial x} = 0, \]
\[ \frac{\partial r_-}{\partial t} + v_- \frac{\partial r_-}{\partial x} = 0, \]
\[ r_+ = u + 2\sqrt{\rho}, \quad r_- = u - 2\sqrt{\rho}, \]
\[ v_+(r_+ - r_-) = \frac{1}{4} (3r_+ + r_-), \quad v_-(r_+ - r_-) = \frac{1}{4} (r_+ + 3r_-). \tag{5} \]

When there is no shock formation in Eqs. (2), the \( \varepsilon \to 0 \) limit is accurately described by the 0-phase solution in Eqs. (4) with slowly varying parameters described by Eq. (5). The shallow water equations are known to admit solutions which "break" – develop an infinite derivative – in finite time. When breaking occurs in Eqs. (5), we require a modification of the 0-phase solution, i.e., an appropriate dispersive regularization.

The dispersive regularization involves slow variations of the NLS 1-phase solution and has the form
\[ \psi_1(\theta_0, \theta_1, x, t) = \sqrt{\rho(\theta_1, x, t)} \times \exp \left( \frac{i}{\hbar} \int_{0}^{x} u(\theta_1(x', t), x', t) \, dx' \right) e^{i\theta_0}, \]
\[
\frac{\partial \theta}{\partial x} = \kappa_0/\varepsilon, \quad \frac{\partial \theta}{\partial t} = -\omega_0/\varepsilon, \quad \frac{\partial \theta}{\partial x} = \kappa_1/\varepsilon, \quad \frac{\partial \theta}{\partial t} = -\omega_1/\varepsilon, \quad \rho(\theta, x, t) = \lambda_3 - (\lambda_3 - \lambda_1) \eta^2(\theta; m), \]
\[
u(\theta, x, t) = V - \frac{\sqrt{\lambda_1 \lambda_2 \lambda_3}}{\rho(\theta, x, t)} \quad m = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}, \quad \sigma = \pm 1, \quad \kappa_1 = \pi \sqrt{\lambda_3 - \lambda_1}/K(m), \quad \omega_1 = \kappa_1 V, \quad \kappa_0 = 0, \quad \omega_0 = \frac{1}{2} (\lambda_1 + \lambda_2 + \lambda_3 - V^2). \tag{6} \]

This is the elliptic function solution of the NLS equation. When the elliptic parameter \( m \to 1 \), this solution converges to the gray soliton solution of the NLS equation \cite{18}. For \( m \to 0 \), this solution corresponds to linear trigonometric waves. Thus, the elliptic function solution traverses linear, small-amplitude behavior to large-amplitude soliton behavior. For \( m \) sufficiently close to 1, we interpret the solution as a soliton train.

There are four independent arbitrary parameters in the elliptic function solution: \( \lambda_1, \lambda_2, \lambda_3, \) and \( V \). With the transformation
\[
\lambda_1 = \frac{1}{16} (r_1 - r_2 - r_3 + r_4)^2, \\
\lambda_2 = \frac{1}{16} (-r_1 + r_2 + r_3 - r_4)^2, \\
\lambda_3 = \frac{1}{16} (-r_1 - r_2 + r_3 + r_4)^2, \tag{7} \\
V = \frac{1}{4} (r_1 + r_2 + r_3 + r_4), \quad r_1 < r_2 < r_3 < r_4, 
\]
and by averaging four of the NLS conservation laws over the fast phase \( \theta_1 \), we obtain the NLS Whitham equations \cite{31, 32}:
\[
\frac{\partial r_i}{\partial t} + v_i(r_1, r_2, r_3, r_4) \frac{\partial r_i}{\partial x} = 0, \quad i = 1, 2, 3, 4. \tag{8a} 
\]

These equations are a system of first order, quasilinear, hyperbolic equations in Riemann invariant form \cite{33}. The 1-phase velocities \( v_i \) are expressions involving complete first \( K(m) \) and second \( E(m) \) elliptic integrals:
\[
 v_1(r_1, r_2, r_3, r_4) = V - \frac{1}{2} (r_2 - r_1) \left[ 1 - \frac{(r_4 - r_2) E(m)}{(r_4 - r_1) K(m)} \right]^{-1}, \\
 v_2(r_1, r_2, r_3, r_4) = V + \frac{1}{2} (r_2 - r_1) \left[ 1 - \frac{(r_3 - r_1) E(m)}{(r_3 - r_2) K(m)} \right]^{-1}, \\
 v_3(r_1, r_2, r_3, r_4) = V - \frac{1}{2} (r_4 - r_3) \left[ 1 - \frac{(r_4 - r_2) E(m)}{(r_4 - r_3) K(m)} \right]^{-1}, \tag{8b} \\
 v_4(r_1, r_2, r_3, r_4) = V + \frac{1}{2} (r_4 - r_3) \left[ 1 - \frac{(r_3 - r_2) E(m)}{(r_4 - r_3) K(m)} \right]^{-1}, \\
 m(r_1, r_2, r_3, r_4) = \frac{(r_4 - r_3)(r_3 - r_2)(r_1 - r_2)(r_1 - r_3)}{(r_4 - r_2)(r_4 - r_3)(r_3 - r_2)}. 
\]

Due to symmetry arguments presented in \cite{24}, if the initial data for Eqs. (8) satisfy
\[
r_1(x, t) = -r_2(-x, t), \quad r_4(x, t) = -r_1(-x, t), \quad t = t_0, \tag{9} 
\]
then the above relations hold for all \( t > t_0 \). This will be helpful during our analysis of the merging problem.

The weak limit in this case is the density and velocity averaged over the period \( 2K(m) \) of the rapidly varying elliptic function \cite{18}:
\[
\bar{\rho} = \lambda_3 - (\lambda_3 - \lambda_1) E(m) \quad \frac{1}{K(m)}, \\
\bar{u} = V - \sigma \sqrt{\lambda_1 \lambda_2 / \lambda_3} - \sigma \sqrt{\lambda_3 - \lambda_1} \times [E(x, 1 - m) + F(x, 1 - m) E(m) / K(m) - 1]], \\
\chi = \sin^{-1} \left( \sqrt{\lambda_3 - \lambda_1} / \lambda_1 \right), 
\]
where \( F(x, 1 - m) \) and \( E(x, 1 - m) \) are the incomplete elliptic integrals of the first and second kinds, respectively.

Given the solution to the Whitham equations (8), the asymptotic solution (6) is constructed by using Eqs. (7) and integrating the phase \( \theta_1 \) in Eqs. (6) appropriately. We take
\[
\theta_1(x, t) = \frac{\pi}{\varepsilon} \int_{x_0}^{x} \sqrt{\lambda_3(x', t) - \lambda_1(x', t)} / [m(x', t)] \, dx' - \int_{x_0}^{t} \nu(x_0, t') \sqrt{\lambda_3(x_0, t') - \lambda_1(x_0, t')} / [m(x_0, t')] \, dt' + \theta_{1,0}, \tag{11} 
\]
where \( \theta_{1,0} \) is the initial phase at \( x = x_0, t = t_0 \).

Because the Jacobi elliptic function \( \text{dn} \) satisfies
\[
\sqrt{1 - m} = \text{dn}(K(m); m) \leq \text{dn}(y; m) \leq \text{dn}(0; m) = 1, 
\]
the oscillations of the density in Eqs. (6) satisfy
\[
\lambda_1 \leq \rho \leq \lambda_2, \tag{12} 
\]
Note that a vacuum point (point of zero density) \cite{19, 23} occurs when
\[
\rho = \lambda_1 = 0. 
\]

2.2. Merging initial value problem

We are interested in the following initial value problem (IVP) for Eq. (1):
\[
\psi(x, 0) = \begin{cases} \sqrt{\rho_0} \exp[-i u_0 x / \varepsilon], & |x| \geq L, \\ 0, & |x| < L, \end{cases} \tag{13} 
\]
where \( \rho_0 \) is the initial density of the left \( (x < -L) \) and right \( (x > L) \) states and \( -\text{sgn}(x) u_0 / \varepsilon \) is the initial phase gradient or superfluid velocity of the left and right states.

The initial data in Eq. (13) result in the following initial data for the density and superfluid momentum equations (2):
\[
\rho(x, 0) = \begin{cases} \rho_0, & |x| \geq L, \\ 0, & |x| < L, \end{cases} \tag{14} 
\]
\[
u(x, 0) = \begin{cases} -\text{sgn}(x) u_0, & |x| \geq L, \\ 0, & |x| < L. \end{cases} \tag{15} 
\]
The velocity is undefined when the density is zero. For definiteness, we take the velocity to be 0 for \( |x| < L \). This IVP is a generalized merging problem for two semi-infinite BECs with uniform initial inward or outward velocities. In order to apply Whitham theory, we map this initial data to the 0-phase Riemann invariants (Eqs. (5)) as
\[
r_{\pm}(x, 0) = \begin{cases} u_0 \pm 2 \sqrt{\rho_0}, & |x| \geq L, \\ 0, & |x| < L. \end{cases} \tag{15} 
\]
As mentioned previously, a similar IVP was considered in \cite{24}. The initial conditions in Eqs. (14) can be obtained from the so-called collision problem involving two counter-propagating DSWs.
Fig. 1. Cases for the merging initial value problem of Eqs. (14). The behavior of the solution depends on the initial velocity \( u_0 \) relative to the initial density \( \rho_0 \). The terms total, partial, and no interaction correspond to the behavior of the two rarefaction waves that propagate from the two initial discontinuities. The labels (a)-(d) correspond to the cases of Figs. 5-7.

In [24] under an appropriate re-scaling and limit. Because of its utility in describing the BEC merging problem and the different dynamics that it engenders, a detailed study of the IVP considered here is useful. Indeed, we note that modulated 0-, 1-, and 2-phase solutions are required to describe the interactions of DSWs whereas here, we will only require modulated 0- and 1-phase solutions. This is a result of the initial zero density separating the two condensates.

As our calculations below will show, a rarefaction wave propagates from each initial discontinuity. Depending on system parameters, these waves may interact leading to an oscillatory interference region.

In general, a system of hyperbolic equations in the small dissipation limit with an initial discontinuity admits two types of wave solutions that depend on the initial data: viscous shock waves and rarefaction waves [34]. This same maxim holds for the dispersive regularization we consider here except that the classical, viscous shock waves are replaced by dispersive shock waves \([18, 19]\). In a fundamental sense, the dispersive hydrodynamics considered here are completely explained by rarefaction wave type solutions to a system of hyperbolic equations. The particular system of hyperbolic equations solved depends on the behavior of the solution. Rarefaction waves in the 0-phase shallow water equations (5) describe the BEC merging problem outside of the oscillatory interference region; i.e., when there is no breaking. Rarefaction wave solutions to the 1-phase Whitham equations (8) describe the interference pattern.

The behavior of the solution depends strongly on the magnitude and sign of the initial velocity \( u_0 \). A summary of the asymptotic behavior is shown in Fig. 1 in the \( u_0-\rho_0 \) plane. For positive velocities (cases (c) and (d)), the two rarefaction waves interact with one another completely, giving rise to a modulated 1-phase region that is bordered by plane waves. For negative velocities sufficiently small in magnitude (case (b)), the two rarefaction waves partially interact, whereas for strongly negative velocities (case (a)), the two rarefaction waves never interact.

The most natural case to consider in the context of a BEC is \( u_0 = 0 \), because dynamic behavior usually starts from the ground state, where the BEC is held steady in a trap potential. We consider this case first in detail to explain our analytical methods, and then we discuss the other cases.

2.3. Interference pattern: \( u_0 = 0 \)

In order to solve the IVP for Eq. (2) with initial data in Eqs. (14) asymptotically \( (0 < \varepsilon \ll 1) \), we invoke a dispersive regularization of Eqs. (2) (see Refs. [23, 24, 33, 35] for detailed discussions of the regularization technique we employ). This regularization is achieved by determining initial data for the 0- and 1-phase Whitham equations, the choice of which is determined by degeneracies or overlapping of two of the four 1-phase Riemann invariants \([r_i]_{i=1}^4 \). The initial data are determined by requiring that (a) the initial data for the averaged density and velocity in Eqs. (10) equal the initial data in Eqs. (14) and (b) the initial data for the Whitham equations give rise to a global solution. When two 1-phase Riemann invariants coincide, they are degenerate, and the two leftover Riemann invariants satisfy the 0-phase equations.

The specific regularization we choose is (see also the top panel in Fig. 2)

\[
\begin{align*}
\rho_1(x,0) &= -2\sqrt{\rho_0} \\
\rho_2(x,0) &= \text{sgn}(x-L)2\sqrt{\rho_0} \\
\rho_3(x,0) &= \text{sgn}(x+L)2\sqrt{\rho_0} \\
\rho_4(x,0) &= 2\sqrt{\rho_0},
\end{align*}
\]

which gives rise to a global solution to Eqs. (8) as we will now show by explicitly solving the Whitham equations (see Fig. 2). By insertion of the initial data (16) into the averaged expressions in Eqs. (10), we can show directly that

\[
\begin{align*}
\rho(x,0) &= \frac{1}{2K(m)} \int_{0}^{2K(m)} \rho(\theta, x, 0) d\theta = \begin{cases} \rho_0 & |x| \geq L, \\ 0 & |x| < L, \end{cases} \\
\pi(x,0) &= \frac{1}{2K(m)} \int_{0}^{2K(m)} u(\theta, x, 0) d\theta \equiv 0,
\end{align*}
\]

and so the dispersive regularization is achieved. Note that this regularization method does not account for slow variation of the phase \( \theta_i \). In particular, if \( \theta_i(0) = \theta_i(x, t) \) in Eq. (11) depends on the slow variables \( x \) and \( t \), this is a higher order effect and cannot be determined within the context of leading order Whitham averaging theory.

Because the initial data in Eqs. (16) satisfy the symmetry conditions of Eqs. (9), the symmetry

\[
r_3(x, t) = -r_2(-x, t)
\]

holds for all \( t \geq 0 \). As such, we will focus on only one of \( r_1 \) or \( r_2 \) in all future discussion, keeping in mind that relation (17) always holds.

Initially, two rarefaction waves develop from the two initial discontinuities at \( x = \pm L \) (see the second panel in Fig. 2). These rarefaction waves are self-similar solutions that satisfy the shallow water equations (5), e.g.,

\[
r'_2[(x-L)/t - v_2(-2\sqrt{\rho_0}, r_2)] = 0,
\]

or

\[
r_2(x,t) = \frac{4}{3t}(x-L) + \frac{2}{3\sqrt{\rho_0}}.
\]

Near \( x = L, r_3 = r_4 \) and hence \( r_2 \) and \( r_4 \) are degenerate so that this leading order 0-phase representation of the solution is valid (see Refs. [23, 24, 33, 35]). Two speeds associated with each rarefaction wave are calculated directly from the 0-phase velocities in Eqs. (5):

\[
\begin{align*}
v_2' &= v_2(r_2) = -2\sqrt{\rho_0}, \quad r_1 = -2\sqrt{\rho_0} = -2\sqrt{\rho_0} \\
v_2'' &= v_2(r_2) = 2\sqrt{\rho_0}, \quad r_1 = -2\sqrt{\rho_0} = 2\sqrt{\rho_0}.
\end{align*}
\]

Interaction of the rarefaction waves occurs by symmetry, at \( x = x_1 = 0 \) at time

\[
L + v_1' t_1 = 0 \Rightarrow t_1 = -\frac{L}{v_1'} = \frac{L}{2\sqrt{\rho_0}}.
\]
In the interaction region of these rarefaction waves, there are no longer any degenerate Riemann invariants. The solution is therefore described by the 1-phase Riemann invariants (see the last three panels in Fig. 2). We label the boundaries of the interaction region \( x_-(t) \) and \( x_+(t) \) for the left and right edges, respectively. We obtain an ordinary differential equation describing these boundaries directly from the 1-phase Whitham equations:

\[
\frac{dy}{ds} = \frac{1}{s} \left( \frac{(y-1)(y-5)}{3(2-y)} \right), \quad y(1) = -1, \tag{20}
\]

where

\[
y(s) = \frac{x_+ \left( \frac{s}{\sqrt{\rho_0}} \right) - L}{L s}, \quad s = \frac{2}{\sqrt{\rho_0}} t. \tag{21}
\]

An algebraic solution to this IVP exists but is cumbersome. Instead, a straightforward phase plane analysis of Eqs. (20) leads to the following statement about the interaction boundary \( x_+(t) \):

\[
0 < x_+(t) < L + \sqrt{\rho_0} t, \quad t > t_1, \\
x_+(t) \sim L + \sqrt{\rho_0} t, \quad t \gg t_1, \\
\sqrt{\rho_0} < \frac{dx_+}{dt} < 2\sqrt{\rho_0}, \quad t > t_1,
\]

with the increase of \( x_+(t) \) and the decrease of \( dx_+/dt \) monotonic. The straight line \( x = L + \sqrt{\rho_0} t \) represents the rightmost, leading edge of the rarefaction wave emanating from \( x = L \). We have therefore shown that the 1-phase interaction region expands to asymptotically \((t \to \infty)\) “catch up” with the expanding rarefaction waves generated initially. A phase diagram in the characteristic \( x-t \) plane (Fig. 3) conveys this behavior.

To calculate \( r_3(x, t) \) for \( t \in (t_1, \infty) \) and \( x_- < x < x_+(t) \), as in the last three panels of Fig. 2, we must solve the symmetric Cauchy problem

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + v_2(2\sqrt{\rho_0}, \rho_0) \frac{\partial \rho}{\partial x} &= 0, \\
\frac{\partial \rho}{\partial t} + v_3(2\sqrt{\rho_0}, \rho_0) \frac{\partial \rho}{\partial y} &= 0, \\
r_2(x_+(t), t) &= \frac{4}{3t} (x_+(t) - L) + \frac{2}{3} \sqrt{\rho_0}, \\
r_3(x_+(t), t) &= 2\sqrt{\rho_0}, \\
r_3(x_-) &= -r_3(x_+(t), t), \quad t > t_1.
\end{align*}
\]

We use a numerical method that integrates along characteristics, similar to that presented in Ref. [29].

### 2.3.1. 0-phase solution

Outside of the 1-phase interaction region, the solution is 0-phase. By direct calculation for \( t < t_1 \), the left rarefaction state is

\[
\Psi_0^{(l)}(x, t) = \left( \frac{2}{3} \sqrt{\rho_0} - \frac{x + L}{3t} \right) \exp \left[ \frac{i}{\epsilon} \left( \frac{(x + L)^2}{3t} + \frac{2}{3} \sqrt{\rho_0} (x + L) - \frac{2}{3} \rho_0 t \right) \right],
\]

\( L - \sqrt{\rho_0} t < x < L + 2\sqrt{\rho_0} t \) (23).

The right rarefaction state is

\[
\Psi_0^{(r)}(x, t) = \left( \frac{2}{3} \sqrt{\rho_0} + \frac{x - L}{3t} \right) \exp \left[ \frac{i}{\epsilon} \left( \frac{(x - L)^2}{3t} - \frac{2}{3} \sqrt{\rho_0} (x - L) - \frac{2}{3} \rho_0 t \right) \right],
\]

\( L - 2\sqrt{\rho_0} t < x < L + \sqrt{\rho_0} t \) (24).

The other 0-phase regions are plane waves of the form \( \Psi = \sqrt{\rho_0} \exp[-\frac{i}{\epsilon} u_0 |x|] \).

### 2.3.2. Linear interference pattern

As previously noted, the left and right rarefaction waves propagate toward one another and interact at the time \( t = t_1 = L/(2\sqrt{\rho_0}) \). The interaction density of these two waves for times sufficiently close to the interaction time \((t = t_1 + \delta t, 0 < \delta t \ll t_1)\) expands to asymptotically \((t \to \infty)\) “catch up” with the expanding rarefaction waves generated initially. A phase diagram in the characteristic \( x-t \) plane (Fig. 3) conveys this behavior.

To calculate \( r_3(x, t) \) for \( t \in (t_1, \infty) \) and \( x_- < x < x_+(t) \), as in the last three panels of Fig. 2, we must solve the symmetric Cauchy problem

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + v_2(2\sqrt{\rho_0}, \rho_0) \frac{\partial \rho}{\partial x} &= 0, \\
\frac{\partial \rho}{\partial t} + v_3(2\sqrt{\rho_0}, \rho_0) \frac{\partial \rho}{\partial y} &= 0, \\
r_2(x_+(t), t) &= \frac{4}{3t} (x_+(t) - L) + \frac{2}{3} \sqrt{\rho_0}, \\
r_3(x_+(t), t) &= 2\sqrt{\rho_0}, \\
r_3(x_-) &= -r_3(x_+(t), t), \quad t > t_1.
\end{align*}
\]
and for $x$ sufficiently close to zero ($|x| \ll L$), can be described by the square modulus of the linear superposition of two modulated counter-propagating plane waves

$$
\rho(x, t) \sim \left| \psi^{(1)}_0(x, t) + \psi^{(R)}_0(x, t) \right|^2
\sim \frac{4 \rho_0 t - 2 \sqrt{\rho_0 x} \exp \left\{ \frac{1}{\varepsilon} \left[ 2 \sqrt{\rho_0 x} - 2 \rho_0 t + \sqrt{\rho_0 L} \right] \right\}}{3L} \times \exp \left\{ \frac{1}{\varepsilon} \left[ -2 \sqrt{\rho_0 x} - 2 \rho_0 t + \sqrt{\rho_0 L} \right] \right\}^2
= 2 \left( \frac{4 \rho_0 t - 2 \sqrt{\rho_0 x}}{3L} \right)^2 \left[ 1 - \cos(4\sqrt{\rho_0 x}/\varepsilon) \right].
$$

This is the well-known linear wave interference pattern.

2.3.3. 1-phase solution

Now that we have determined the dynamics of the 0- and 1-phase solution parameters, we reconstruct the modulated 0- and 1-phase asymptotic solution via Eqs. (4), (6) and (11) in Fig. 4. The left column of panels corresponds to the density $\rho$, and the right column represents the velocity $u$. Two rarefaction waves propagating through the vacuum eventually interact, giving rise to a modulated oscillatory region that expands in time to become a soliton train.

Some properties of the asymptotic solution can be noted immediately. By symmetry, we know that $r_1(0, t) = -r_2(0, t)$ for $t \geq 0$; hence $\lambda_1(0, t) = V(0, t) = 0$, and the phase, assuming $\rho_0 = 0$ and $t_0 = t_1$ in Eq. (11), can be written

$$
\theta_1(x, t) = \frac{\pi}{\varepsilon} \int_0^x \frac{\sqrt{\lambda_1(x', t) - \lambda_1(x, t)}}{K(m(x', t))} \, dx' + \theta_{1,0},
$$

and takes the value $\theta_{1}(0, t) = \theta_{1,0}$ at the origin.

The initial data in Eq. (13) for $\rho$ and $u$ satisfy the symmetry properties in Eq. (3). Thus, we require that $\rho$ be even in $x$ and $u$ be odd in $x$ for all time. Based on the symmetry property (17), we can directly use Eqs. (7) and (26) to show that

$$
\theta_1(-x, t) = -\theta_1(x, t)
$$

if and only if

$$
\theta_{1,0}(-x, t) = -\theta_{1,0}(x, t).
$$

Given this result, the only way for $\rho$ and $u$ to be even and odd, respectively, is to require that $\theta_{1,0}(0, t) = 0$. Then the density at the origin is $\rho(\theta_{1,0}(0, t), 0, t) = 0$; i.e. a vacuum point. This corresponds to a dark soliton pinned at the origin. Numerical simulations of the 1D NLS equation suggest that the density $\rho(\theta_{1,0}(0, t), 0, t)$ is at a maximum. A maximum density at the origin occurs if $\theta_{1,0}(0, t) = K(m(0, t))$ but breaks the even/odd symmetry in $\rho/\mu$ of the asymptotic solution we have constructed. The correct determination of the phase is the subject of future work.

From the parameter dynamics in Fig. 2, we see that $r_2(x, t) \to r_3(x, t)$ for $x \cdot t = x \cdot t_1$, and $1 \ll t < O(1)/\varepsilon$. Using a small $r_3 - r_2$ expansion for the asymptotic solution (6), we find that

$$
\rho(\theta_{1,0}(x, t), 0, t) = \rho_0 - \frac{1}{4} [4 \rho_0 - r_3^2(x, t)] \text{sech}^2(\theta_{1,0}(x, t)) + O(r_3 - r_2).
$$

The interaction region degenerates to a dark soliton. This can also be seen by expanding the oscillation wavelength

$$
l = \frac{2K(m)\varepsilon}{\sqrt{\lambda_3 - \lambda_1}} = \frac{-2\varepsilon}{\sqrt{4\rho_0 - r_3^2}} \ln((r_3 - r_2)\sqrt{\rho_0}) + O(1) \to \infty, \quad \text{as} \quad r_2 \to r_3,
$$

which grows as the difference $r_3 - r_2$ decreases, as shown in Fig. 4. The oscillation amplitude is

$$
A = \lambda_2 - \lambda_1,
$$

$$
= \rho_0 - \frac{1}{4} r_3^2 + \frac{1}{4} (-2\sqrt{\rho_0} + r_3)(r_3 - r_2) + O((r_3 - r_2)^2).
$$

For comparison with the linear interference pattern in Eq. (25), we now consider the behavior of the 1-phase asymptotic solution for times $\delta t = t - t_1 > 0$ small. The 1-phase Whitham solution to the merging problem in this region can be written

$$
r_1 = -2\sqrt{\rho_0} = -r_4, \quad r_2 = -2\sqrt{\rho_0} + v_2(x, t),
$$

$$
r_3 = 2\sqrt{\rho_0} - v_3(x, t), \quad 0 < v_2, v_3 \ll 2\sqrt{\rho_0}.
$$

In this case, the modulated 1-phase elliptic function density in Eqs. (6) takes the small amplitude form

$$
\rho(\theta_{1,0}(x, t), x, t) \sim \frac{1}{16} \left[ v_2(x, t)^2 + v_3(x, t)^2 \right]
- \frac{1}{2} v_2(x, t)v_3(x, t) \cos(2\theta_{1,0}),
$$

$$
\theta_{1,0} \sim \frac{2\sqrt{\rho_0 x}}{\varepsilon} + \theta_{1,0}.
$$

This density matches the linear interference pattern in Eq. (25) if

$$
v_2 \sim v_3 \sim \frac{16}{3L} [4 \rho_0 t - 2 \sqrt{\rho_0 x}], \quad \theta_{1,0} = 0.
$$

Eqs. (25), (27) and (29) show that the elliptic function solution we have calculated encapsulates (a) the small density, trigonometric interference pattern during the initial stages of interaction, and (b) the soliton train for longer times.
2.4. Interference pattern: $u_0 \neq 0$

In this section, we briefly discuss the solution to the IVP (14) for $u_0 \neq 0$. Recall from Fig. 1 that the behavior of the asymptotic solution depends on the relationship between the velocity $u_0$ and density $\rho_0$. The initial data considered here correspond to the situation where the initially separated BECs have an inward ($u_0 > 0$) or outward ($u_0 < 0$) superfluid velocity. This could be realized experimentally, for example, by phase imprinting [36,37].

A representative phase diagram in the characteristic x-t plane is shown in Fig. 5(a)–(d) for each case shown in Fig. 1. Corresponding solutions of the Whitham or shallow water equations and their associated 0- or 1-phase asymptotic solutions at specific times (the horizontal dashed lines in Fig. 5(a)–(d)) are shown in Figs. 6(a)–(d) and Figs. 7(a)–(d), respectively. These asymptotic solutions demonstrate how physically different the dynamics of the merging problem can be from the case $u_0 = 0$ studied in the previous subsection.

2.4.1. $u_0 < -2\sqrt{\rho_0}$

When $u_0 \leq -2\sqrt{\rho_0}$, the two initial rarefaction waves propagate away from one another, so there is no 1-phase interaction (see Fig. 5(a), Fig. 6(a), and Fig. 7(a)). The asymptotic solution is described, leading order in $\epsilon$, by the classical 0-phase result for the shallow water equations (5).

2.4.2. $-2\sqrt{\rho_0} < u_0 < 0$

When $-2\sqrt{\rho_0} < u_0 < 0$, interaction of the rarefaction waves begins at the time

$$t_1 = \frac{L}{u_0 + 2\sqrt{\rho_0}},$$

and leads to a modulated 1-phase region, as shown in Fig. 5(b)–(d), Fig. 6(b)–(d), and Fig. 7(b)–(d). The particular case $-2\sqrt{\rho_0} < u_0 < 0$ (Fig. 5(b), Fig. 6(b), Fig. 7(b)) has similar behavior to the $u_0 = 0$ case studied extensively in Section 2.3 except that the edges of the interaction region do not asymptotically ($t \gg t_1$) coincide with the leading rarefaction wave edge.

2.4.3. $0 < u_0 < 2\sqrt{\rho_0}$

A phase diagram in the characteristic x-t plane for the case $0 < u_0 < 2\sqrt{\rho_0}$ is shown in Fig. 5(c). Here, we see that the right edge of the interaction boundary intersects the right edge of the rarefaction wave propagating from the initial $x = L$ discontinuity. This leads to a change in the internal structure of the interference pattern as shown in Fig. 7(c).
2.4.4. \(2\sqrt{\rho_0} < u_0\)

The last case considered corresponds to a large inward velocity. The structure of the asymptotic solution is similar to the previous case \(0 < u_0 < 2\sqrt{\rho_0}\) except for one important difference. The characteristics internal to the interaction region (represented by the nonhorizontal dashed curves in Fig. 5(d)) cross. After the crossing of these characteristics, the Whitham solution is constant and the density is locally periodic, oscillating between the maximum and minimum values \(\rho_1 = 0 \leq \rho(\theta(t,x,t),x,t) < 4\rho_0 = \rho_2\), independent of the superfuid velocity \(u_0\). Interestingly, a locally periodic wave that oscillates between the vacuum and the maximum value \(4\rho_0\) occurs in two other distinct settings: the dispersive Riemann problem with large initial velocity difference [19] and the piston DSW problem with large piston velocity [38].

3. The merging problem: 3D Gross–Pitaevskii equation

In this section, we briefly show how our result for the 1D NLS equation in Section 2.3 can be used to describe the interference pattern generated when two BECs merge.

3.1. Gross–Pitaevskii equation

For the merging problem we consider here, the dispersive equation that describes a BEC in 3D is [25–27]:

\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2\epsilon} \Delta \Psi + V(x,y,z,t)\Psi + |\Psi|^2 \Psi,
\]

\[
V(x,y,z,t) = \frac{1}{2}(x^2 + \kappa^2 z^2) + H(-t) \frac{A}{d_t d_z} e^{-((x/d_t)^2+|z|/d_z)^2} \times \int_{\mathbb{R}^3} |\Psi|^2 d^3x = 1, \quad r = \sqrt{x^2 + z^2},
\]

\[
\epsilon = \left(\frac{\hbar}{m_0 (4\pi a_s N)^{1/3}}\right)^{1/5} \ll 1, \quad \kappa = \frac{\omega_{yz}}{\omega_x},
\]

where \(\epsilon\) is the small dispersion parameter, \(N\) is the number of condensed particles, \(m\) is the atomic mass, \(a_s\) is the scattering length, \(\hbar\) is Planck’s constant divided by 2\(\pi\), \(\omega_x\) is the long \(x\)-axis harmonic trapping frequency, \(\omega_{yz}\) is the harmonic trapping frequency along the \(y\) and \(z\) axes, \(H\) is the Heaviside step function modeling the rapid turning off of a dipole laser beam, \(A\) is proportional to the dipole laser power, \(d_t = w_t/\sqrt{2}\) is the normalized laser waist along the \(x\)-axis, \(d_z = w_z/\sqrt{2}\) is the normalized laser waist along the \(z\)-axis, and \(l\) is the nonlinear length scale of the axial condensate (proportional to the Thomas–Fermi radius) defined below. The nondimensionalization we used to obtain this equation from dimensional, primed coordinates is

\[
t = \omega_x t', \quad (x,y,z) = (x', y', z')/l, \quad \Psi = l^{1/2} \Psi',
\]

\[
l = \left(\frac{4\pi \hbar^3}{m^2 a_s N}\right)^{1/5}.
\]

For times \(t < 0\), the potential in Eqs. (31) consists of two parts: the Gaussian repulsive dipole laser beam and the magnetic harmonic trap. The Gaussian potential models a dipole laser beam propagating in the \(y\) direction that is repulsive because \(A > 0\). The harmonic trap is highly anisotropic and often called cigar shaped because the confinement in the radial \(y\) and \(z\) direction is much stronger, \(\kappa \gg 1\), than in the \(x\) direction. Such a trap has been shown, under certain conditions, to lead to effective 1D behavior where the tight confinement effectively “freezes” out any excitations along the radial direction (e.g., see Refs. [39–42] and references therein). We take advantage of this behavior in order to describe the 3D condensate by our 1D results of Section 2.

The experimental parameters that we use are \(N = 20,000\) rubidium atoms, \(m = 1.45 \times 10^{-25}\) kg, \(a_s = 5.5\) nm, \((\omega_x, \omega_{yz})/(2\pi) = (7402)\) Hz, the dipole laser power \(P = 150\) mW, \(w_t = 8\) \(\mu\)m, and \(w_z = 32\) m\(\mu\)m. These lead to the following values of the nondimensional and normalization parameters: \(\epsilon = 0.097, \quad \kappa = 57.4, A = (1.0 \times 10^8)^{-1}\)\(\mu\)l, \(d_t = 0.434, d_z = 1.74\), and \(l = 13.0\) m\(\mu\)m.

Throughout this work, we assume that the wavefunction \(\Psi\) is in the ground state for the harmonic trap and dipole laser potentials when \(t < 0\). Using the numerical method discussed in [43], we compute the 3D ground state of Eqs. (31) in the form \(\Psi(x, y, z, t) = e^{i(\phi(x, y, z, t) - \phi(x, y, z, 0))}\). We are interested in the dynamics of \(\Psi\) when \(t > 0\) and the dipole laser potential is turned off. This is the BEC merging problem.

3.2. One-dimensional reduction

In Ref. [39], an effective 1D GP equation was derived to describe the dynamics of a BEC in a cigar shaped trap. We follow [39] in what follows.

For the experiments considered here, the laser waist in the \(z\) direction is much larger than the radial extent of the condensate: \(w_z = 32\) \(\mu\)m \(\gg 2\) \(\mu\)m \(\approx\) BEC radial extent.

Our 3D numerical simulations of Eqs. (31) discussed below, show that the condensate maintains effective azimuthal symmetry along the tightly confined directions. As such, we neglect the \(z\) variation in the dipole laser beam in our 1D reduction and assume that \(V\) in Eq. (31) is given by

\[
V(x, r, t) = \frac{1}{2}(x^2 + \kappa^2 z^2) + H(-t) \frac{A}{d_t d_z} e^{-((x/d_t)^2+|z|/d_z)^2}.
\]

We assume that the wavefunction \(\Psi\) can be decomposed into a product of the radial harmonic oscillator ground state and an effective 1D wavefunction,

\[
\Psi(x, r, t) = e^{-i\phi/\sqrt{\kappa}} \phi(x, t).
\]

After insertion of this ansatz into Eqs. (31), integration over the radial direction, and with the rescaling

\[
\tilde{x} = x/d_z, \quad \tilde{t} = t/T, \quad T = \frac{\pi \epsilon d_z}{\kappa}, \quad \phi(x, t) = \frac{\pi \epsilon d_z}{\kappa},
\]

we find the following effective 1D equation for the merging problem,

\[
i\hbar \frac{\partial \tilde{\phi}}{\partial \tilde{t}} = -\frac{\hbar^2}{2\epsilon} \tilde{\phi} + \left[\sigma^2 \tilde{\phi}^2 + B(\tilde{x})\right] \tilde{\phi} + |\tilde{\phi}|^2 \tilde{\phi},
\]

\[
\int_{-\infty}^{\infty} |\tilde{\phi}(\tilde{x}, \tilde{t})|^2 d\tilde{x} = 1, \quad \tilde{\phi} = \frac{\pi \epsilon d_z}{\kappa}, \quad B(\tilde{x}) = \beta \epsilon e^{-\tilde{x}^2}, \quad \beta = \frac{2\pi \epsilon A}{\kappa d_z}, \quad \sigma = \frac{\pi \epsilon d_z}{\kappa}.
\]

With the experimental parameters mentioned earlier, we find \(\tilde{\epsilon} = 0.015, T = 0.029, \sqrt{\pi \epsilon d_z} / \kappa = 0.048, \beta = 0.093,\) and \(\sigma = 0.021\). This normalization reveals three important length scales:

<table>
<thead>
<tr>
<th>Length Scale</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>trap length</td>
<td>(O(\sigma) \gg 1)</td>
</tr>
<tr>
<td>oscillation</td>
<td>(O(\tilde{x}) \ll 1)</td>
</tr>
<tr>
<td>dipole laser waist</td>
<td>(O(1))</td>
</tr>
</tbody>
</table>

We take advantage of these multiple scales to apply the results of Section 2 to Eq. (33). To this end, we may simplify Eq. (33) further by modeling the Gaussian dipole laser potential with the box function

\[
B(\tilde{x}) = \beta H(\tilde{x} - 1).
\]
This assumption leads to a convenient, asymptotic form for the ground state of Eqs. (33).

For $t < 0$, we assume that the wavefunction in Eqs. (33) with $B(x)$ defined in Eq. (35), is in the quasi-stationary state $\phi(x, t) = e^{-i\mu t}/g(x)$ with $g(x) \in \mathbb{R}$. This leads to the following nonlinear eigenvalue problem

$$\mu g = -\frac{\epsilon^2}{2} g_{xx} + \left[ \sigma^2 x^2 + \beta H(|x|) - 1 \right] g + g^3.$$ 

Neglecting the second derivative term because $0 < \epsilon \ll 1$, we find the Thomas–Fermi state

$$g(x) \sim \begin{cases} \sqrt{\mu - \sigma^2 x^2} & 1 \leq |x| \leq \sqrt{\frac{\mu}{\sigma}} \\ 0 & \text{otherwise.} \end{cases} \quad (36)$$

The eigenvalue $\mu$ is determined from the normalization condition in Eqs. (33),

$$\int_{-\infty}^{\infty} |g(x)|^2 \, dx = 1 \Rightarrow \mu^{3/2} - \mu - \frac{3\sigma}{2} + \frac{2\sigma^3 - 3\sigma}{4} = 0.$$ 

3.3. Asymptotic solution to the 3D merging problem

The well separated length scales in (34) suggest that we can represent the asymptotic solution to the 3D BEC merging problem as the 0- and 1-phase Whitham solutions (Eqs. (4) and (6)) to the initial value problem in Eqs. (2) and (14) with $L = 1$, modulated by the slowly varying background $g(x)/\sqrt{\rho_0}$, normalized to unity in the region where the dipole beam is initially applied. Such an assumption is valid because the rapidly varying oscillations depending on $\theta_1 = O(1/\epsilon) \gg 1$, $\sigma = O(1)$, and their slow modulations depending on $\tilde{x}, \tilde{t} = O(1)$ all vary more rapidly than the initial condition, the ground state in Eq. (36), which depends on $\sigma |x| = O(\sigma) \ll 1$. Then this solution is valid so long as $|x| \ll 1/\sigma$.

4. Experiments and numerical simulations

In this section we compare new experimental results for the BEC merging problem with full 3D numerical simulations of Eqs. (31) and our asymptotic result of Sections 2.3 and 3.

4.1. Experiments

The experiments begin with about 20,000 $^{87}$Rb atoms in the $|F, m_f = 1, -1 \rangle$ hyperfine state. The atoms are magnetically confined in an elongated Ioffe–Pritchard type trap with the same trap frequencies as given in Section 3.1. For a BEC in the ground state, this implies a ratio of $\mu/\hbar \omega_k \approx 2.4$, where $\mu$ is the chemical potential and $\hbar \omega_k$ is the radial oscillator level spacing. Therefore our experiment approaches the quasi-1D regime. A repulsive barrier for the atoms is created with a dipole laser at a wavelength of 660 nm, that is far detuned from the Rb D-lines at 780 nm and 795 nm. The dipole beam parameters are the same as those mentioned in Section 3.1. The dipole laser is sent horizontally through the center of the magnetic trap, along the radial (tightly confining) $y$-direction creating a repulsive barrier. By rapidly turning off the repulsive dipole beam within less than 250 ns, dynamics such as those shown in Fig. 8 are induced. To enlarge the resulting features, we employ an anti-trapped expansion 2 ms long before imaging [44].

The domain simulated was $156 \times 4 \times 4 \times 4 \mu m$ with 2048 $\times$ 64 $\times$ 64 grid points. No anti-trapped expansion was performed. Fig. 9 shows the integrated density $\int_{-\infty}^{\infty} |\Psi(x, y, z, t)|^2 \, dz$ from (a) 3D simulations and the (b) asymptotic solution. These simulation results compare well with the experimental results of Fig. 8. The asymptotic solution qualitatively captures the behavior of the merging problem. The structure of the asymptotic solution density in Fig. 9(b) is narrower and longer than the 3D simulation of Fig. 9(a) and the oscillation wavelengths differ slightly. Nonetheless, given all the assumptions we have made in deriving the asymptotic solution, the qualitative agreement is strong enough to support our claim that the interference pattern due to the merging of two BECs eventually develops into a modulated soliton train.

To further demonstrate the transition from linear, trigonometric wave interference patterns to soliton trains, consider Fig. 10. These are magnified versions of $y = 0, z = 0$ slices of the 3D numerical simulations in the second and sixth panels of Fig. 9(a). Early during the interaction process of the two BECs, the interference pattern is essentially trigonometric as shown in the left panel of Fig. 10. In the right panel, the interference pattern, after a sufficient evolution time, has developed into a soliton train just as our asymptotic solution predicts (see discussion in Sections 2.3.2 and 2.3.3).

5. Conclusion

We have solved the BEC merging problem asymptotically, in the small dispersion limit, and numerically, with favorable comparison to new experimental results for the cigar shaped trap geometry. The results show that matter wave interference between two BECs of sufficiently large density can be interpreted as a modulated soliton train resulting from the interaction of two rarefaction waves propagating in the vacuum. Furthermore, the soliton...
train was shown to emerge from a small density, trigonometric interference pattern generated early during the interaction process. Right: soliton train after sufficient merging evolution. Compare these with the left 3rd and 5th panels of Fig. 4.

Acknowledgments

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References


Fig. 9. (a) The integrated density of the 3D numerical simulations for the merging experiments depicted in Fig. 8. (b) The integrated density of the 3D asymptotic solution to the merging experiment.

Fig. 10. Magnified versions of $y = 0, z = 0$ slices of the 3D numerical simulations. Left: trigonometric interference pattern generated early during the interaction process. Right: soliton train after sufficient merging evolution. Compare these with the left 3rd and 5th panels of Fig. 4.