```
> restart;
  with(plots):
  with(plottools):
  with(LinearAlgebra):
```

Zeeman Catastrophe Machine

Assume the bed is fixed to a circle of radius r (scale to 1 by choosing length scale)
Two springs (with zero natural length) and spring constant k (scale to 1 by choosing time scale)
One fixed at (-a,0) and the other at a variable point (x,y)

See the paper:
Litherland, T. J. and A. Siahmakoun (1995). “Chaotic behavior of the Zeeman Catastrophe Machine. ” Am. J. Phys. 63: 426-431.

Set up:
Measure theta from the postive x-axis.
Define the positions of the bob, and the two spring endpoints

```
> bob:= r*<cos(theta),sin(theta)>;
  spring1:= <-a,0>;
  spring2:= <x,y>;
  bobPt:= convert(bob,list):
  s1Pt := convert(spring1,list):
  s2Pt:= convert(spring2,list):
```

$$bob := \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$$

$$spring1 := \begin{bmatrix} -a \\ 0 \end{bmatrix}$$

$$spring2 := \begin{bmatrix} x \\ y \end{bmatrix} \tag{1.1}$$

Spring forces = -k *(Length-Resting Length)*(unit vector along spring)
Let the resting length be L, and the spring lengths be d1 and d2

```
> F1:= -k*(d1-L)/d1*(bob -spring1);
  F2:= -k*(d2-L)/d2*(bob-spring2);
```

$$F1 := \begin{bmatrix} -\frac{k\,(d1-L)\,(r\cos(\theta)+a)}{d1} \\ -\frac{k\,(d1-L)\,r\sin(\theta)}{d1} \end{bmatrix}$$

$$F2 := \begin{bmatrix} -\frac{k\,(d2-L)\,(r\cos(\theta)-x)}{d2} \\ -\frac{k\,(d2-L)\,(r\sin(\theta)-y)}{d2} \end{bmatrix} \tag{1.2}$$

The lengths of the springs

```
> d1:= norm(bob-spring1,2, conjugate =false);
  d2:= norm(bob-spring2,2,conjugate=false);
```

$$d1 := \sqrt{(r\cos(\theta)+a)^2 + r^2\sin(\theta)^2}$$

$$d2 := \sqrt{(r\cos(\theta)-x)^2 + (r\sin(\theta)-y)^2} \tag{1.3}$$

Motion is allowed only in the tangent direction

```
> t:= <-sin(theta),cos(theta)>;
```

$$t := \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \tag{1.4}$$

Net tangential force is

```
> F:= unapply(simplify(DotProduct(t,F1+F2,conjugate=false)),[x,y]);
```

$$\begin{aligned} F := (x,y) \rightarrow -\Big(k\Big(& -\sin(\theta)\sqrt{r^2-2r\cos(\theta)x+x^2-2r\sin(\theta)y+y^2}\sqrt{r^2+2r\cos(\theta)a+a^2}\,a \\ & +\sin(\theta)\sqrt{r^2-2r\cos(\theta)x+x^2-2r\sin(\theta)y+y^2}\,L\,a \\ & +\sin(\theta)\sqrt{r^2+2r\cos(\theta)a+a^2}\sqrt{r^2-2r\cos(\theta)x+x^2-2r\sin(\theta)y+y^2}\,x \\ & -\sin(\theta)\sqrt{r^2+2r\cos(\theta)a+a^2}\,L\,x \\ & -\cos(\theta)\sqrt{r^2+2r\cos(\theta)a+a^2}\sqrt{r^2-2r\cos(\theta)x+x^2-2r\sin(\theta)y+y^2}\,y \\ & +\cos(\theta)\sqrt{r^2+2r\cos(\theta)a+a^2}\,L\,y\Big)\Big) \Big/ \\ & \left(\sqrt{r^2+2r\cos(\theta)a+a^2}\sqrt{r^2-2r\cos(\theta)x+x^2-2r\sin(\theta)y+y^2}\right) \end{aligned} \tag{1.5}$$

Choose some values

```
> L:=1.0;
  a:= 2:
  x0:=2:
  r:=1:
  y0:= -0.5:
  k:=1:
```

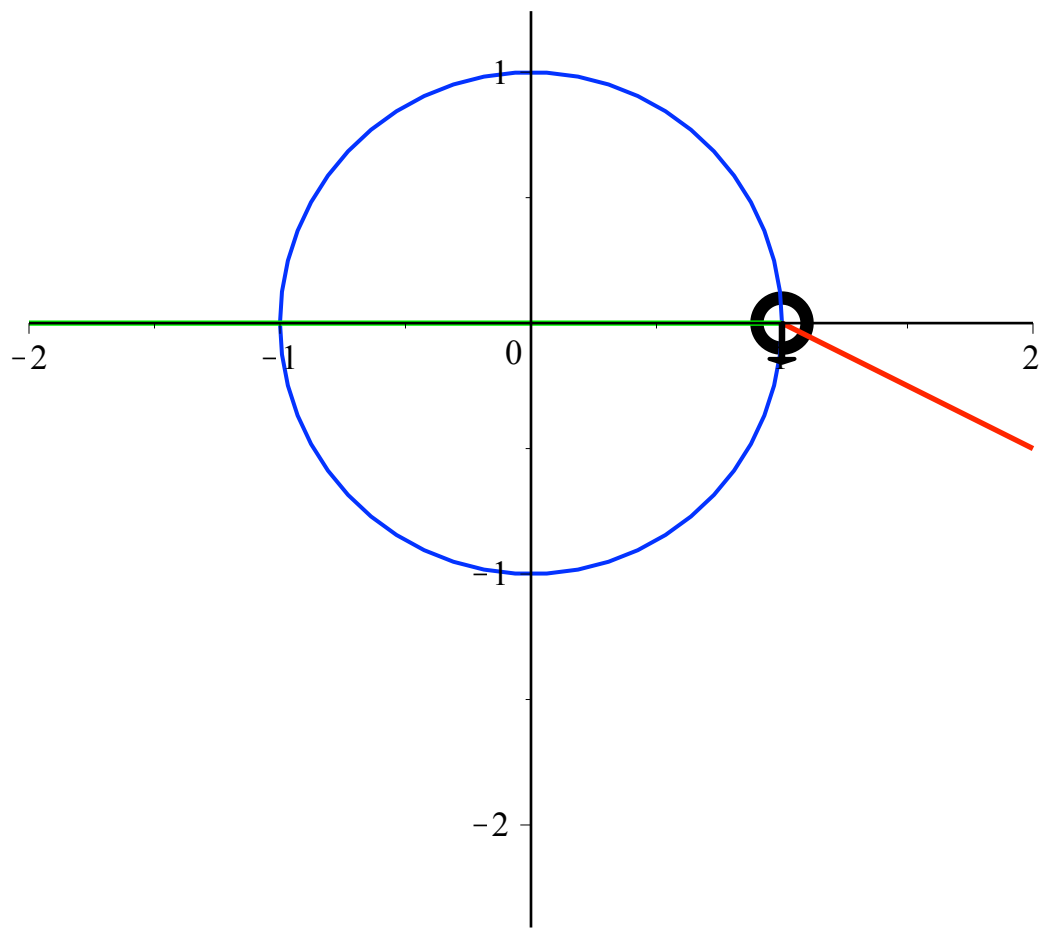
$$L := 1.0 \tag{1.6}$$

```
> simplify(F(x,y));
```

$$\begin{aligned} \Big(1.\Big(& 2.\sin(\theta)\sqrt{1.-2.x\cos(\theta)+x^2-2.y\sin(\theta)+y^2}\sqrt{5.+4.\cos(\theta)} \\ & -2.\sin(\theta)\sqrt{1.-2.x\cos(\theta)+x^2-2.y\sin(\theta)+y^2} \\ & -1.\sin(\theta)\sqrt{5.+4.\cos(\theta)}\sqrt{1.-2.x\cos(\theta)+x^2-2.y\sin(\theta)+y^2}\,x \\ & +\sin(\theta)\sqrt{5.+4.\cos(\theta)}\,x \\ & +\cos(\theta)\sqrt{5.+4.\cos(\theta)}\sqrt{1.-2.x\cos(\theta)+x^2-2.y\sin(\theta)+y^2}\,y \\ & -1.\cos(\theta)\sqrt{5.+4.\cos(\theta)}\,y\Big)\Big) \Big/ \\ & \left(\sqrt{1.-2.x\cos(\theta)+x^2-2.y\sin(\theta)+y^2}\sqrt{5.+4.\cos(\theta)}\right) \end{aligned} \tag{1.7}$$

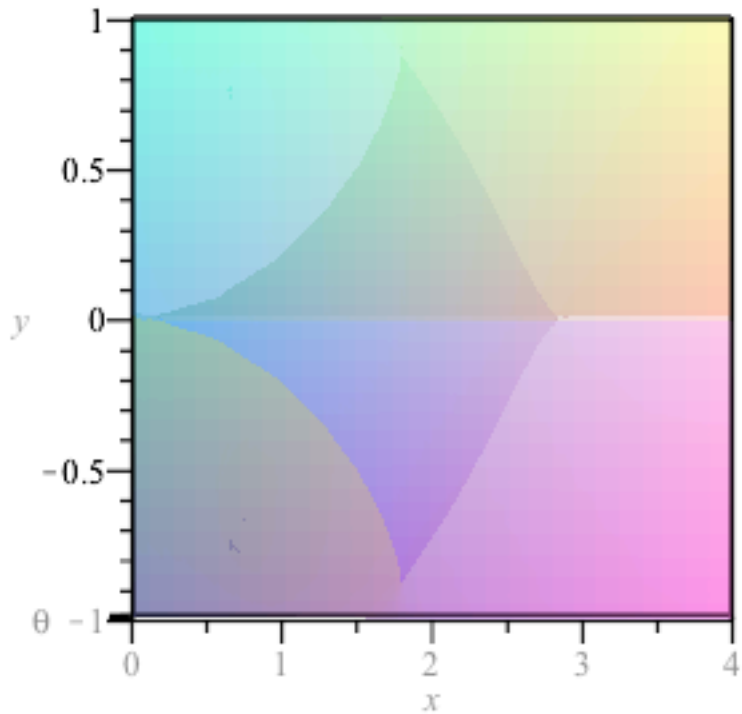
```
> Fvect:= 3*F(x0,y0)*t:
  c:= circle([0,0],1,color=blue):
  b:= circle(bobPt,0.1,color=black, thickness=5):
  s1:=line(s1Pt,bobPt, color=green, thickness =2):
  s2:=line([x0,y0],bobPt,color=red, thickness = 2):
  arr:= arrow(bob,Fvect,0.01,0.1,0.1,color=black):
  animate(display,[b,c,s1,s2,arr,scaling=constrained],theta=0..2*
  Pi, frames=50);
```

$\theta = 0.$



Bifurcation set

```
> implicitplot3d(F(x,y),x=0..4,y=-1..1,theta=-Pi..Pi, style=
  surface,transparency=0.0,  grid=[50,50,50],axes=boxed);
```



Analytical solution?

```
> DF:=diff(F,theta);
  simplify(%);
> solve(%,theta);
```