APPM 5460 J.D. Meiss

# Dynamics Projects

You task is to find a topic that interests you and to read about it. The final project is not intended to be original research; it is fine to present what other people have done, as long as you give them the proper credit and write your paper in your own words. The purpose is to learn, in some depth, some ideas and applications of Dynamical Systems. Your project might have a computational component, but this is not necessary. In your paper, you should explain the details of the calculations (typically a journal article will leave out many of the details). It is essential that your project deal with dynamical systems modeled by ordinary differential equations, and that it applies or goes beyond the material that we cover in class. It is also important that you not just quote from a textbook: base your project around a journal article. You may need to supplement the article with textbook material, of course, but the article should be primary.

## Rules of the Game

- You should work in a group of two. An **ideal group** contains one Engineering student and one APPM student. Note: there are 13 students of each type in the class.*
- You should let me know about the topic and the title of your project by **Feb 26**. I would prefer to have at most one group working on each topic, though we can arrange to divide a topic into parts if two groups are interested in it.
- A 2-4-page project proposal, with literature review will be due by **March 23**. The purpose of this is to get you working on the project well before it is due!
- The final project will be presented as a 10-15 minute presentation during the last week of class or at the final exam period (May 5, 4:30-7:00 PM)
- The written project will be due the last day of class (**May 2**).

## Possible Projects

All of these projects involve some library research. Some tools that will help you:

- The MathSciNet web site http://ams.rice.edu/mathscinet/search has a very useful online reference list.
- The ISI Web of Knowledge can allow you to find modern articles that refer to a classic paper. See http://apps.isiknowledge.com/.
- The “Frequently Asked Questions” document for the newsgroup sci.nonlinear includes a list of software and references, see http://amath.colorado.edu/faculty/jdm/faq.html.
- The DSWeb site is a good place to look for interesting things about dynamics, see http://www.dynamicalsystems.org

The list below includes some suggested projects. You are by no means required to choose one of these. However, any topic you do choose must be **approved**.

### Dynamical Systems approach to Electric Circuits

It is possible to view the equations of an electric circuit as a system of ODE’s, recall §1.4. Some techniques of dynamical systems can be applied to the study of these equations. In particular, circuits for power supplies and rectifiers can be studied using Lyapunov functions and multiple-scale analysis; the goal is not always to find chaos, but in certain cases, to show stability of the system. On other cases, such as the “famous” Chua circuit, will have chaotic behavior (Matsumoto, Chua et al. 1984). Some references are given in the FAQ for sci.nonlinear.

### Phase Space Reconstruction

Describe the Takens theory of phase space embedding based on the Whitney embedding theorem and how it can be used in experiments to show the presence or absence of chaos. For references see (Ott, Sauer et al. 1994) and (Wolf, Swift et al. 1985).

* If you are a loner, you *can* work alone. However, your presentation might have to be shorter, since there are 26 students.

**Forecasting**

Nonlinear Forecasting, as introduced in (Farmer and Sidorowich 1987), is a hot topic (this paper has been cited over 900 times!). How does it work (it usually involves phase space reconstruction, see above)? Apply it to some representative data. What are the fundamental limitations on prediction implied by chaos? See e.g. Sugihara and May (Sugihara and May 1990) and the introduction to (Letellier, Moroz et al. 2008).

**Dynamical Systems approach to Economics**

Explain some possible applications of Dynamical Systems to Economics. In particular, discuss why some applications to Microeconomics seem to fail. Explain why the so-called "Invisible Hand" might be an idea that has no reality. This corresponds to the belief that an equilibrium allocation (zero excess demand) can be viewed as a stable fixed-point solution of a suitable dynamical system (with the introduction of a tatonnement). A good reference is (Saari 1995).

**Economic cycles**

Is there an economic cycle? Why do we have periods of prosperity and periods of recession? It is not clear that economic cycles really exist (Laaksonen and Matti 1996, Szydowski, Krawiec et al. 2001).

**Models of economic growth**

There are many models of economic growth, generally resulting from dynamic optimization that can be written in terms of a system of differential equations. For instance, one can study the models of Solow and Ramsey (Chiang 1992, Shone 1997, Klein 1998).

**Chaos and Finance**

There are a number of interesting and controversial applications of dynamical systems to the research of capital markets. Some of the authors claim to have new points of view on how to deal with the prediction of stock prices. At the center of the controversy is the question of the validity of the Efficient Market Hypothesis, and the existence of positive Lyapunov exponents in the historical stock data. The book (Peters 1996) might be useful.

**Kolmogorov-Arnold-Moser (KAM) Theorem**

Kolmogorov proposed in 1954 a remarkable stability result for conservative (Hamiltonian) Dynamical systems. Basically, it says that such systems when weakly perturbed away from a case do not immediately become completely chaotic. This implies the possible existence, at the same time, of regular and chaotic behavior. Explain the KAM theorem, discuss its history, and the consequences for the ``ergodic hypothesis" of Boltzmann. (The proof is long and difficult, requiring lots of advanced mathematics). See (de la Llave 2001, Pöschel 2001)

**Chemical Pattern Modeling**

There have been interesting experiments recently on patterns arising from simple chemical reactions. Could these explain the Leopard's spots and the Zebra's stripes? See the article (Swinney 1993) and book (Murray 1993).

**Biological Modeling**

Dynamical systems have been successful in Biology; recall the examples in Ch. 1. Discuss some biological models for disease propagation, population dynamics, DNA replication, etc. See the book (Murray 1993). Another interesting area is virology: (Nowak and May 2001). A recent paper has studied the effect of a "minimal infectious dose" in the propagation of cholera (Joh, Wang et al. 2009). Another recent study shows that three populations have interesting chaotic behavior (Previte and Hoffman 2013).

**Modeling of Chaotic Toys**

Develop and investigate a mathematical model for a chaotic toy such as

- Double Pendulum (Shinbrot, Grebogi et al. 1992).
- Magnetic Pendulum (like Wildwood pendulum in my office!).
- Driven Pendulum (Starrett and Tagg 1995)
- Sprung Pendulum (Rusbridge 1979, Alasty and Shabani 2006).

- The Levitron (Berry 1996, Dullin and Easton 1999) (a challenging one to model!)

**Modeling the Triple Pendulum in the ITLL**

The triple-pendulum chaos demonstration in the lobby of the Integrated Teaching and Learning Lab (on the east side of the engineering center) has been instrumented. You can take data corresponding to the position of all of the rotating joints. Your task would be to take some data and compare it to the solutions of a model for the system. Phase space embedding may prove useful (Sauer, Yorke et al. 1991).

**Synchronization of Oscillators**

Coupled oscillators provide models for diverse phenomena from Josephson junctions to the flashing of fireflies. What is remarkable is that a system with many degrees of freedom sometimes gets into a synchronized state where all the oscillators move in unison. Why does this occur and what are the requirements on the oscillators and the coupling? See (Strogatz 2000), (Kopell and Ermentrout 1988) and (Pecora, Carroll et al. 2000).

**Mixing**

The problem of mixing has to do with many Industrial applications, see §1.4. How is mixing described in dynamics? Describe mixing and ergodic theory. Think about some possible industrial applications. Give a possible relation of this to Fluid Dynamics. This field was begun in the paper (Aref 1984); see the book (Ottino 1990) and (Wiggins and Ottino 2004). For a recent review see (Haller 2013).

**Poincaré's solution to the three-body problem**

The story of Poincaré and how he gave a solution to the Restricted Three Body Problem is the story of a contest that was fixed, so that he could win it, see p. 161 of DDS and (Diacu and Holmes 1996). This project involves some historical research, but fortunately, two recent papers (Barrow-Green 1997) (Andersson 1994) have everything that you want to know about this interesting story.

**Tests for Chaos**

Most numerical tests for chaos use Lyapunov exponents (see Ch. 7), but this is notoriously expensive. Alternative ideas have to do with looking at short-period, unstable periodic orbits to extrapolate the exponents (Cvitanovic 1995) or to compute the fractal dimension of an attractor (Viswanath 2004). The idea of "finite-time Lyapunov exponents" or "fast Lyapunov indicators" is often used, especially for nonautonomous systems (Haller and Yuan 2000, Paleari, Froeschle et al. 2008). A sophisticated, recent idea is the so-called *0-1 Test for Chaos* (Gottwald and Melbourne 2003, Gottwald and Melbourne 2009). Study one of these methods or compare them!

**Chaos in the Solar System**

Some people claim that they have found some chaotic behavior in the solar system. What are the consequences for us? Is the claim true? What are the bases for their investigations? How did they do all this? Read papers by J. Laskar, e.g. (Laskar 1996), and J. Wisdom, e.g. (Sussman and Wisdom 1992).

**The *N*-Body Problem**

Describe some current advances in the 3,4, and 5-body problems. In particular, describe the possibility of a finite time singularity in the 5-body problem. You should show a general understanding of the *N*-body problem. See (Saari and Xia 1995). Another interesting area is the recent discovery of an exotic "figure-eight" orbit in the *n*-body problem, see (Montgomery 2001, Chenciner, Gerver et al. 2002).

**Central Configurations**

Some special cases of the *N*-body problem are extremely regular. In fact, there are configurations that never change their relative shape. These are called central configurations and are important in the study of the general N-body problem. There are many open questions to be answered. For example, the total number central configurations for 5 bodies is not known. Understand the definition of Central Configuration and write a program that finds them for different values of the parameters. To start, you can read (Moeckel 1990). An interesting case is Maxwell's ring, see (Barrabés, Cors et al. 2012).

**Hard vs Soft Chaos**

For many years it was believed that some simple systems describing the motion of a particle in a potential well where completely chaotic, indeed Gutzwiller called these systems "hard chaos." More recently some of the systems, for example the Hamiltonian with an $x^2y^2$ potential have been shown to have stable periodic orbits, see (Dahlqvist and Russberg 1990). Why were these orbits missed for so long and how were they found?

**Hilbert's 16th Problem**

Hilbert conjectured that the number of limit cycles for a polynomial differential equation on the plane is finite. Remarkably this is still an open conjecture, though partial results have been obtained. Look up some of these results and investigate models on the computer determining the number of limit cycles numerically. Some references to this are §6.6, see also (Shi 1988) and (Ilyashenko and Yakovenko 1995).

**Control of Chaos**

Chaos exhibits the surprising feature that it is quite "controllable"—indeed the sensitive dependence can be exploited to give a large effect for a small controller. A famous study by Ott, Grebogi and Yorke started this field (Ott, Grebogi et al. 1990), but many studies have taken off from there, e.g. (Ding, Ott et al. 1994, Starrett and Tagg 1995, Li, Wen et al. 2001).

**Piecewise Smooth Systems**

Much of the analysis that we do in class only works when the ODEs are smooth, and yet there are many systems, particularly in engineering (for example systems with impacts like the clattering of a train wheel on a track) for which this assumption breaks down. Lots of new research has been done in this area and the state of the art circa 2006 is summarized in (di Bernardo, Budd et al. 2007) and see the more recent review (Makarenkov and Lamb 2012)

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