Stochastic Ordering Based Carrier-to-Interference Ratio Analysis for the Shotgun Cellular Systems

Prasanna Madhusudhanan, Juan G. Restrepo, Youjian (Eugene) Liu, Timothy X. Brown, and Kenneth R. Baker

***Abstract*—An analytical tool based on usual stochastic ordering is developed to compare the distributions of carrier-to-interference ratio at the mobile station of cellular systems where the base stations are distributed randomly according to certain non-homogeneous Poisson point processes. The comparison is done by studying the base station densities without having to solve for the distributions of the carrier-to-interference ratio, which are often hard to obtain.**

***Index Terms*—Carrier-to-interference ratio, co-channel interference, fading channels, stochastic ordering.**

I. Introduction

The Poisson point process has been extensively adopted as a model for spatial arrangement of transmitter nodes in the study of cellular, ad-hoc, and other uncoordinated and decentralized communication networks [1]–[4, and references therein]. Throughout this paper, transmitters are referred to as base stations (BSs). An underlying assumption of these works is that the BS density is constant, i.e., the Poisson point process is homogeneous. Such a model does not sufficiently represent the often heterogeneous distribution of BSs in reality.

As a result, in [2], we have modeled the BS arrangement by non-homogeneous Poisson point processes in $\mathbb{R}^l$, $l = 1, 2, \text{ and } 3$, and derived semi-analytical expressions for the distribution of carrier-to-interference ratio $\left(\frac{C}{I}\right)$ at a given mobile station (MS). This helped in quantifying the outage probability in a very general setting involving arbitrary fading distribution and path-loss models, random BS transmission powers and user mobility. However, in the absence of results in closed form, alternate approaches need to be explored to develop a clearer understanding of such networks. Furthermore, growing interests in understanding networks like femtocells, cognitive radios and heterogeneous networks have created a demand to consider more complex stochastic geometric models, with little hope for complete analytical characterization.

Here, we pose the question: Is it possible to qualitatively compare two $\frac{C}{I}$ distributions by only examining the BS densities without having to obtain the $\frac{C}{I}$ distributions? This paper answers the question affirmatively for certain BS densities by developing a usual stochastic ordering based tool.

Concepts of stochastic ordering have been previously applied to scenarios of interest in wireless communications. Recently, in [5], performances of communication systems are explored solely using stochastic ordering. Further, [6] focuses on the study of directionally convex ordering of shot-noise fields, and establishes the ordering of interferences in networks with nodes distributed according to different point processes. This paper focuses on establishing usual stochastic ordering between the $\frac{C}{I}$ at the MS in non-homogeneous Poisson point process based BS arrangements, where the MS connects to the strongest BS, which is different than the setup considered in [5], [6]. More examples on multi-tier networks and cognitive radios can be found in [3], [4].

The next section gives the system model. The main result of this paper is Theorem 1 in Section III. The utility of this result is explored in Section IV, by considering several scenarios of interest in wireless networks. Section V concludes the paper.

II. System Model

The *Shotgun Cellular System (SCS)* is a model for the cellular system in which the BSs are distributed in a l-dimensional plane ($l-$D, typically $l = 1, 2, \text{ and } 3$) according to a non-homogeneous Poisson point process in $\mathbb{R}^l$. The intensity function of the Poisson point process is called the BS density function.

Without loss of generality, we restrict our attention to 1-D SCSs, because for the $\frac{C}{I}$ analysis, the l-D SCSs can be reduced to an equivalent 1-D SCS [2, Lemma 2] with a BS density function $\lambda(r)$, where $r \geq 0$ is the distance of the BS from a mobile-station (MS) located at the origin. For example, a *homogeneous* l-D SCS with density $\lambda_0\,(> 0)$ is equivalent to a 1-D SCS with density function $\lambda\,(r) = \lambda_0 b_l r^{l-1},\ r \geq 0,\ b_1 = 2,\ b_2 = 2\pi, \text{ and } b_3 = 4\pi$ [2, Corollary 2].

The BSs are assumed to have independent and identically distributed (i.i.d.) random transmission powers K_i's and shadow fadings Ψ_i's across BSs. The path-loss is $R^{-\varepsilon}$, $\varepsilon > 0$, where R denotes the distance between the BS and the receiver. We assume an interference limited system and omit thermal noise. We focus on the signal quality of a MS at the origin. The MS chooses to communicate with the BS, referred to as the "serving BS," that corresponds to the *strongest* received signal power, or equivalently strongest $\frac{C}{I}$. All other BSs are the "interfering BSs". The signal quality at the MS is measured by $\frac{C}{I} = \frac{K_S \Psi_S R_S^{-\varepsilon}}{\sum_{i=1}^{\infty} K_i \Psi_i R_i^{-\varepsilon}}$, where S indexes the serving BS, i indexes the interfering BSs and the random variables $R_S \leq R_1 \leq R_2 \leq \cdots$ are ordered BS locations.

III. The Stochastic Ordering of $\frac{C}{I}$

In this section, we present the theoretical tools that are used to compare $\frac{C}{I}$ tail probability by comparing the equivalent 1-D BS densities. Since the effect of i.i.d. shadow fading factors

Manuscript received June 20, 2012. The associate editor coordinating the review of this letter and approving it for publication was O. Dabeer.

P. Madhusudhanan, Y. Liu, and T. X. Brown are with the Department of Electrical, Computer and Energy Engineering Department; J. G. Restrepo is with the Department of Applied Mathematics; and T. X. Brown and K. R. Baker are with the Interdisciplinary Telecommunications Program, University of Colorado, Boulder, CO 80309-0425 USA (e-mail: {mprasanna, juanga, youjian.liu, timxb, kenneth.baker}@colorado.edu).

Digital Object Identifier 10.1109/WCL.2012.080112.120458

and i.i.d. transmission powers can be captured by modifying the BS density as shown in Section IV-D, they are assumed to be 1 for all BSs. The generalization to arbitrary path loss model is given in [2, Section VI], which is also equivalent to modifying the BS density $\lambda(r)$. As a result, $\frac{C}{I} = \frac{R_1^{-\varepsilon}}{\sum_{i=2}^{\infty} R_i^{-\varepsilon}}$.

Definition 1. Let X and Y be two random variables such that $\mathbb{P}(\{X > x\}) \leq \mathbb{P}(\{Y > x\}),\ \forall\ x \in (-\infty, \infty)$, then X is *smaller than* Y *in the usual stochastic order* and this is denoted by $X \leq_{\text{st}} Y$. Further, $X =_{\text{st}} Y$ means $\mathbb{P}(\{X > x\}) = \mathbb{P}(\{Y > x\}),\ \forall\ x \in (-\infty, \infty)$. [7, p. 3]

If X and Y are the $\frac{C}{I}$ at the MS in two different SCSs, $X \leq_{\text{st}} Y$ implies that the MS in the SCS corresponding to Y is more likely to achieve better signal quality than in the SCS corresponding to X. Let $\{R_k\}_{k=1}^{\infty}$ represent the set of distances of BSs from the MS, indexed in the ascending order of the distance, and let $D_{k+1} = R_{k+1} - R_k$ be the distance between two adjacent BSs, and $f_{D_{k+1}|R_k}(d|\, r; \lambda(s))$ be the probability density function (p.d.f.) of D_{k+1} conditioned on $R_k = r$, as a function of the BS density $\lambda(s)$.

Lemma 1.

$$f_{D_{k+1}|R_k}(d\,|r; \lambda(s)) \overset{(A)}{=} e^{-\int_r^{r+d} \lambda(s)ds} \lambda(r+d), \text{ and}$$

$$f_{aD_{k+1}|aR_k}(d'\,|r'; \lambda(s)) \overset{(B)}{=} f_{D_{k+1}|R_k}\left(d' \middle| r'; \frac{1}{a}\lambda\left(\frac{s}{a}\right)\right).$$

Proof: Equation (A) follows from the properties of Poisson processes [8], [9]. Equation (B) is proved by $f_{aD_{k+1}|aR_k}(d'|r'; \lambda(s)) \overset{(a)}{=} \frac{1}{a} f_{D_{k+1}|R_k}\left(\frac{d'}{a} | \frac{r'}{a}; \lambda(s)\right) \overset{(b)}{=} \frac{1}{a}\lambda\left(\frac{r'+d'}{a}\right)\exp\left(-\int_{\frac{r'}{a}}^{\frac{r'+d'}{a}} \lambda(s)ds\right) \overset{(c)}{=} \frac{1}{a}\lambda\left(\frac{r'+d'}{a}\right)\exp\left(-\int_{r'}^{r'+d'} \frac{1}{a}\lambda\left(\frac{s'}{a}\right)ds'\right)$, where (a) is obtained by a variable change; (b) follows from (A); and (c) is obtained by a variable change and gives (B). ■

Lemma 1 means that scaling D_{k+1} and R_k by a is equivalent to scaling the BS density as $\frac{1}{a}\lambda(\frac{r}{a})$. The significance of Lemma 1 is presented next, using the notation defined below.

Definition 2. $\frac{C}{I}\big|_{\lambda(r)}$ denotes the random variable of the $\frac{C}{I}$ at the MS of a 1-D SCS with BS density function $\lambda(r)$.

Corollary 1. $\frac{C}{I}\big|_{\lambda(r)} =_{\text{st}} \frac{C}{I}\big|_{\frac{1}{a}\lambda(\frac{r}{a})},\ \forall\ a > 0.$

Proof: Let $\{R_k\}_{k=1}^{\infty}$ correspond to the 1-D SCS with BS density function $\lambda(r)$. Then, since the ordered BS locations R_k's are determined by inter-BS distances, it follows from Lemma 1 that $\frac{C}{I}\big|_{\lambda(r)} = \frac{(aR_1)^{-\varepsilon}}{\sum_{k=2}^{\infty}(aR_k)^{-\varepsilon}}\Big|_{\lambda(r)} =_{\text{st}} \frac{\left(R_1'\right)^{-\varepsilon}}{\sum_{k=2}^{\infty}\left(R_k'\right)^{-\varepsilon}}\Bigg|_{\frac{1}{a}\lambda(\frac{r}{a})}$, where R_k''s corresponding to $\frac{1}{a}\lambda(\frac{r}{a})$ have the same distribution as aR_k's corresponding to $\lambda(r)$. ■

The following special case is a direct corollary of the above result.

Corollary 2. In a *homogeneous* l-D SCS, $\frac{C}{I}$ is not a function of the BS density.

Proof: Firstly, recall that the $\frac{C}{I}$ at the MS in a *homogeneous* $l-$D SCS with BS density λ_0 is the same as that in a $1-$D SCS with a BS density function $\lambda(r) = \lambda_0 b_l r^{l-1}$. Next, from Corollary 1, the distribution of $\frac{C}{I}$ in this SCS is the same as that in a 1-D SCS with the BS density function $\frac{1}{a}\lambda\left(\frac{r}{a}\right) = \lambda_0 \alpha b_l r^{l-1},\ \alpha = a^{-l},\ a > 0$. Thus, distributions of $\frac{C}{I}$ corresponding to $\alpha\lambda_0$ and λ_0 are the same. ■

The above result was also observed in [2], but Corollary 2 provides a simpler and more fundamental proof. Next, we define a notation used in Theorem 1.

Definition 3. For BS density function $\lambda(r)$, the cumulative BS density function is defined as $\mu(r) \triangleq \int_0^r \lambda(s)ds$, and its inverse function is defined as $\mu^{-1}(q) \triangleq \sup\{r : \mu(r) \leq q\}$.

Since $\lambda(r) \geq 0$, $\mu(r)$ is a non-decreasing function of r. In general, the inverse function is not injective since $\lambda(r)$ can be zero in arbitrary intervals of $r \in [0, \infty)$. The above definition makes it injective. For certain BS densities, it is possible to compare two $\frac{C}{I}$'s by comparing the densities without solving for the distributions. This is facilitated by Theorem 1.

Theorem 1. Let $\left\{\lambda_1(r), \mu_1(r), \mu_1^{-1}(q)\right\}$ and $\left\{\lambda_2(r), \mu_2(r), \mu_2^{-1}(q)\right\}$ be the BS density functions, cumulative BS density functions and their inverse functions for two 1-D SCSs, respectively. The $\frac{C}{I}$ at the MS follows the usual stochastic order $\frac{C}{I}\big|_{\lambda_1(r)} \leq_{\text{st}} \frac{C}{I}\big|_{\lambda_2(r)}$, if for each $q > 0$ and $a = \frac{\mu_2^{-1}(q)}{\mu_1^{-1}(q)}$, $\frac{1}{a}\lambda_1\left(\frac{r}{a}\right) \geq \lambda_2(r),\ \forall\ r \geq \mu_2^{-1}(q)$.

See Appendix for the proof.

Outage probability $P\{\frac{C}{I+N} < \gamma\}$ and ergodic rate $\mathbb{E}\left[\log\left(1 + \frac{C}{I+N}\right)\right]$ are often used to measure the performance for slowly varying and fast varying systems, respectively. For interference dominated systems, if $\frac{C}{I}\big|_{\lambda_1(r)} \leq_{st} \frac{C}{I}\big|_{\lambda_2(r)}$, it is obvious that $P\left\{\frac{C}{I}\big|_{\lambda_1(r)} < \gamma\right\} \geq P\left\{\frac{C}{I}\big|_{\lambda_2(r)} < \gamma\right\}, \forall\gamma$. In addition, for $u\left(\frac{C}{I}\right) = \log\left(1 + \frac{C}{I}\right)$ or any other non-decreasing function, $\mathbb{E}\left[u\left(\frac{C}{I}\big|_{\lambda_1(r)}\right)\right] \leq \mathbb{E}\left[u\left(\frac{C}{I}\big|_{\lambda_2(r)}\right)\right]$. Applications of the above theorem are in the next section.

IV. Applications of the $\frac{C}{I}$ Stochastic Ordering

A. *Comparison of Homogeneous* $l-$*D SCSs* ($l = 1, 2,$ and 3)

Here, we show that the signal quality degrades as the dimension l of the *homogeneous* l-D SCS increases, for which we need the following corollaries.

Corollary 3. For each $q > 0$ and $a = \frac{\mu_2^{-1}(q)}{\mu_1^{-1}(q)}$, if $\Delta(r) \triangleq \frac{1}{a}\lambda_1(\frac{r}{a}) - \lambda_2(r)$ is a non-decreasing function for all $r \geq 0$, then $\frac{C}{I}\big|_{\lambda_1(r)} \leq_{\text{st}} \frac{C}{I}\big|_{\lambda_2(r)}$.

Proof: Note that $\int_0^{\mu_2^{-1}(q)} \frac{1}{a}\lambda_1\left(\frac{s}{a}\right)ds = q = \int_0^{\mu_2^{-1}(q)} \lambda_2(s)ds$. Hence, $\int_0^{\mu_2^{-1}(q)} \Delta(s)ds = 0$. Suppose $\Delta(\mu_2^{-1}(q)) < 0$, then $\Delta(r) < 0,\ r \in [0, \mu_2^{-1}(q)]$, since $\Delta(r)$ is non-decreasing. This is a contradiction. Thus, $\Delta(\mu_2^{-1}(q)) \geq 0$. Using Theorem 1, the corollary is proved. ■

Corollary 4. For a *homogeneous* l-D SCS with BS density λ_0 and its equivalent 1-D BS density function $\lambda_l(r) = \lambda_0 b_l r^{l-1},\ r \geq 0$, multiplying $\lambda_l(r)$ with a non-increasing

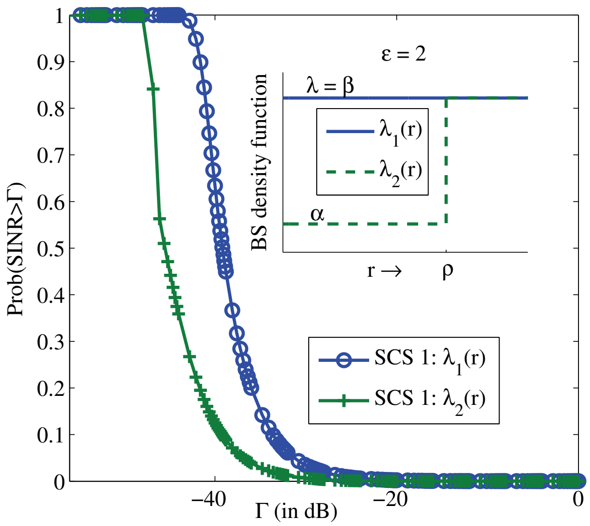

Fig. 1. Comparison of $\frac{C}{I}$ tail probability of two 1-D SCSs.

function $\beta(r) > 0$ improves the $\frac{C}{I}$, i.e., $\frac{C}{I}\big|_{\lambda_l(r)} \leq_{\rm st} \frac{C}{I}\big|_{\beta(r)\lambda_l(r)}$. The inequality reverses if $\beta(r)$ is non-decreasing.

Proof: If $\beta(r)$ is non-increasing, for any $a > 0$, the density difference $\Delta(r) = \frac{1}{a}\lambda_l(\frac{r}{a}) - \beta(r)\lambda_l(r) = \lambda_0 b_l\left(\frac{1}{a^{l-2}} - \beta(r)\right) r^{l-1}$ is non-decreasing. By Corollary 3, $\frac{C}{I}\big|_{\lambda_l(r)} \leq_{\rm st} \frac{C}{I}\big|_{\beta(r)\lambda_l(r)}$ holds. If $\beta(r)$ is non-decreasing, the same proof applies with $\Delta(r) = \beta(r)\lambda_l(r) - a\lambda_l(\frac{r}{a})$. ∎

Hence, $\frac{C}{I}\big|_{\lambda_1(r)} \overset{(a)}{\geq}_{\rm st} \frac{C}{I}\big|_{\lambda_2(r)} \overset{(b)}{\geq}_{\rm st} \frac{C}{I}\big|_{\lambda_3(r)}$ by plugging $l = 1, 2, 3$ in $\lambda_l(r)$, respectively; (a) holds because $\lambda_2(r) = \beta(r)\lambda_1(r)$, where $\beta(r) = \frac{b_2}{b_1}r$ is a non-decreasing function; and similarly (b) also holds. Thus, the comparison between $\frac{C}{I}$s is done without finding their distributions.

B. A Qualitative Comparison between Two 1-D SCSs

Consider a *homogeneous* 1-D SCS with a BS density function $\lambda_1(r) = \lambda,\ r \geq 0$, and another 1-D SCS with a BS density function $\lambda_2(r) = \begin{cases}\alpha & 0 \leq r \leq \rho \\ \beta & r > \rho\end{cases}$, where $\alpha < \beta$ (see inset of Figure 1). Such $\lambda_2(r)$ might describe, for example, a highway passing through a region of small population (BS density of α) and then a region of greater population (BS density of β). Usually, such a scenario is approximated by a constant BS density throughout the highway, which is represented by $\lambda_1(r)$. Corollary 4 gives the intuitive result $\frac{C}{I}\big|_{\lambda_1(r)} \geq_{\rm st} \frac{C}{I}\big|_{\lambda_2(r)}$ and puts the intuition on solid foundation. Further, Figure 1 shows this stochastic ordering using simulations, the steps for which can be found in [2, Appendix D]. On the other hand, if $\alpha > \beta$, $\frac{C}{I}\big|_{\lambda_1(r)} \leq_{\rm st} \frac{C}{I}\big|_{\lambda_2(r)}$. Using Corollary 4, similar scenarios can be studied for 2-D and 3-D.

C. Comparison of Path-loss Models

Here, we compare the $\frac{C}{I}$ at the MS in two 1-D SCSs with BS density functions $\lambda_i(r)$ and path-loss models $\frac{1}{h_i(r)}, \forall\, r \geq 0$, for $i \in \{1, 2\}$. Assume the derivative $h_i'(r)$ exists and $h_i'(r) > 0\ \forall\ r > 0$. In the proof of the following corollaries, we use [2, Theorem 4] to reduce the 1-D SCS with BS density function $\lambda_i(r)$ and path-loss model $\frac{1}{h_i(r)}$ to an equivalent 1-D SCS with BS density function $\bar{\lambda}_i(r) = \begin{cases}\frac{\lambda_i\left(h_i^{-1}(r)\right)}{h_i'\left(h_i^{-1}(r)\right)} & r \geq h_i(0) \\ 0 & r < h_i(0)\end{cases}$ and a path-loss model, $\frac{1}{r}$, where $h_i^{-1}(\cdot)$ is the inverse function. Although $\frac{1}{r}$ is singular at the origin, this conversion works for any non-singular path-loss models $\frac{1}{h(r)}$, such as $\frac{1}{h(r)} = \frac{1}{1+r^{\varepsilon}},\ \varepsilon > l$, for which $\bar{\lambda}(r) = 0$ for $0 \leq r \leq h(0) = 1$, avoiding the singular point.

Corollary 5. In a *homogeneous* $l-$D SCS, if the path-loss follows a power-law parameterized by a path-loss exponent, ε, the $\frac{C}{I}$ at the MS improves as the path-loss exponent increases. In other words, if $h_i(r) = r^{\varepsilon_i},\ i = 1, 2$, such that $\varepsilon_1 > \varepsilon_2 > l$, then $\left(\frac{C}{I}\right)_1 \geq_{\rm st} \left(\frac{C}{I}\right)_2$, where $\left(\frac{C}{I}\right)_i$ corresponds to the path-loss model $\frac{1}{h_i(r)}$.

Proof: Using [2, Theorem 4], the equivalent 1-D SCSs with a path-loss model $\frac{1}{r}$ have the BS density functions $\bar{\lambda}_i(r) = \frac{\lambda_0 b_l}{\varepsilon_i} r^{\frac{l}{\varepsilon_i}-1}$. Further, $\bar{\lambda}_2(r) = \beta(r)\bar{\lambda}_1(r)$, where $\beta(r) = \frac{\varepsilon_1}{\varepsilon_2} r^{\frac{l}{\varepsilon_2}-\frac{l}{\varepsilon_1}},\ r \geq 0$ is a non-decreasing function. Hence, Corollary 4 applies and $\frac{C}{I}\big|_{\bar{\lambda}_1(r)} \geq_{\rm st} \frac{C}{I}\big|_{\bar{\lambda}_2(r)}$. ∎

Hence, a simple proof that does not require solving the distribution of $\frac{C}{I}$ gives the expected result that a system with a greater path-loss exponent has a better $\frac{C}{I}$. The following corollary establishes a similar result between two popularly used path-loss models [10].

Corollary 6. In a *homogeneous* $l-$D SCS with a BS density λ_0, the received signal of a MS located at the origin satisfies $\left(\frac{C}{I}\right)_1 \leq_{\rm st} \left(\frac{C}{I}\right)_2$, where $\left(\frac{C}{I}\right)_1$ corresponds to path-loss $\frac{1}{h_1(r)}$ with $h_1(r) = r^{\varepsilon_1},\ r \geq 0$ and $\left(\frac{C}{I}\right)_2$ corresponds to the path-loss $\frac{1}{h_2(r)}$ with $h_2(r) = \begin{cases} r^{\varepsilon_1} & , r \leq 1 \\ r^{\varepsilon_2} & , r > 1\end{cases}$, where $\varepsilon_2 > \varepsilon_1 > l$. The opposite conclusion holds when $\varepsilon_1 > \varepsilon_2 > l$.

Proof: Using [2, Theorem 4], $\bar{\lambda}_1(r) = \frac{\lambda_0 b_l}{\varepsilon_1} r^{\frac{l}{\varepsilon_1}-1},\ r \geq 0$, and $\bar{\lambda}_2(r)$ satisfies the equation $\bar{\lambda}_2(r)\beta(r) = \bar{\lambda}_1(r)$, where $\beta(r) = \begin{cases}1 & , r \leq 1 \\ \frac{\varepsilon_1}{\varepsilon_2} r^{\frac{l}{\varepsilon_2}-\frac{l}{\varepsilon_1}} & , r > 1\end{cases}$. Since $\varepsilon_2 > \varepsilon_1 > l$, $\beta(r)$ is a non increasing function. As a result, Corollary 4 holds and hence $\frac{C}{I}\big|_{\bar{\lambda}_1(r)} \leq_{\rm st} \frac{C}{I}\big|_{\bar{\lambda}_2(r)}$. Thus, the system with the path-loss model $\frac{1}{h_2(r)}$ has a better signal quality compared to that of $\frac{1}{h_1(r)}$. When $\varepsilon_1 > \varepsilon_2 > l$, $\beta(r)$ is a non decreasing function and $\frac{C}{I}\big|_{\bar{\lambda}_1(r)} \geq_{\rm st} \frac{C}{I}\big|_{\bar{\lambda}_2(r)}$. ∎

D. Shadow Fading and Random Transmission Powers

Shadow fading, random transmission powers, and arbitrary path-loss models together can be captured by modifying the BS density function as follows [2, Theorem 3, 4]. Consider a 1-D SCS with BS density function $\lambda(r)$, path-loss model $\frac{1}{h(r)}$, random shadow fading factors $\{\Psi_i\}_{i=1}^{\infty}$, and transmission powers $\{K_i\}_{i=1}^{\infty}$. Ψ_i and K_i can be correlated but are i.i.d. across i. For the $\frac{C}{I}$ analysis, the SCS is equivalent to another 1-D SCS with BS density function $\bar{\lambda}(r) = \mathbb{E}_{\Psi,K}\left[K\Psi\tilde{\lambda}(K\Psi r)\right]$, where $\tilde{\lambda}(r) =$

$\begin{cases} \frac{\lambda\left(h^{-1}(r)\right)}{h'\left(h^{-1}(r)\right)} & r \geq h(0) \\ 0 & r < h(0) \end{cases}$; $\mathbb{E}$ is the expectation operator w.r.t. Ψ and K; $\Psi =_{\mathrm{st}} \Psi_i$ and $K =_{\mathrm{st}} K_i, \ \forall\ i$. This holds as long as the expectation converges.

Thus, after modifying the BS density function, Theorem 1 applies. The following corollary shows a scenario where $\frac{C}{I}$ distribution is unaffected by shadow fading and random transmission powers.

Corollary 7. In a *homogeneous* $l-$D SCS with a BS density λ_0, and a path-loss model $\frac{1}{r^{\varepsilon}},\ \varepsilon > l$, the $\frac{C}{I}$ distribution at the MS does not depend on the random shadow fading factors $\{\Psi_i\}_{i=1}^{\infty}$ and transmission powers $\{K_i\}_{i=1}^{\infty}$, if they are i.i.d. across BSs and $\left|\mathbb{E}_{\Psi,K}\left[(\Psi K)^{\frac{l}{\varepsilon}}\right]\right| < \infty$.

Proof: The equivalent 1-D density function $\lambda(r) = \lambda_0 b_l r^{l-1}$, and the path-loss model $h(r) = r^{\varepsilon}$ gives $\tilde{\lambda}(r) = \frac{\lambda_0 b_l}{\varepsilon} r^{\frac{l}{\varepsilon}-1}$. We have $\left(\frac{C}{I}\right)_{\frac{1}{r^{\varepsilon}},\text{random } \Psi_i, K_i, \lambda(r)} \overset{(a)}{=}_{\mathrm{st}} \frac{C}{I}\Big|_{\frac{1}{r},\Psi_i=K_i=1,\bar{\lambda}(r)} \overset{(b)}{=}_{\mathrm{st}} \frac{C}{I}\Big|_{\frac{1}{r},\Psi_i=K_i=1,\frac{1}{\alpha}\tilde{\lambda}\left(\frac{r}{\alpha}\right)} \overset{(c)}{=}_{\mathrm{st}} \frac{C}{I}\Big|_{\frac{1}{r},\Psi_i=K_i=1,\tilde{\lambda}(r)}$, where (a) is obtained with $\bar{\lambda}(r) = \mathbb{E}_{\Psi,K}\left[\Psi K \tilde{\lambda}(\Psi K r)\right] = \mathbb{E}_{\Psi,K}\left[(\Psi K)^{\frac{l}{\varepsilon}}\right]\tilde{\lambda}(r)$; (b) is obtained by rewriting $\bar{\lambda}(r)$ as $\frac{1}{\alpha}\tilde{\lambda}\left(\frac{r}{\alpha}\right)$ where $\alpha = \left(\mathbb{E}_{\Psi,K}\left[(\Psi K)^{\frac{l}{\varepsilon}}\right]\right)^{-\frac{\varepsilon}{l}}$; (c) is obtained by applying Corollary 1. Thus, the $\frac{C}{I}$ distribution is unaffected by shadow fading and transmission powers. ∎

This result was already proved in [2, Remark 4(a)]. Here, however, we have shown an alternative proof that is based only on the concepts of usual stochastic ordering.

V. Conclusions

This paper is an extension to our work in characterizing the $\frac{C}{I}$ of SCS in [2]. We have developed a tool based on stochastic ordering to compare the $\frac{C}{I}$ at the MS in non-homogeneous Poisson processes. With Theorem 1, we show that, by just comparing certain BS density functions of the SCSs, we can make strong inferences such as, a MS in a given SCS achieves a $\frac{C}{I}$ that is at least as good as that achieved in another SCS without having to solve for the $\frac{C}{I}$ distributions. Moreover, as a consequence of Theorem 1, simple proofs are given to show that in a homogeneous l-D SCS, (1) a MS sees decreasing signal quality as dimension l increases; (2) the $\frac{C}{I}$ at the MS improves as the path-loss exponent of the channel increases; and (3) $\frac{C}{I}$ distribution is unaffected by shadow fading and random transmission powers.

Appendix

Consider the 1-D SCS specified by the set $\left\{\lambda(r), \mu(r), \mu^{-1}(q)\right\}$, as in Definition 3. The following remark relates $\frac{C}{I}$ to the cumulative BS density.

If R_1 denotes the distance between the serving BS and MS in the 1-D SCS, $\mathbb{P}\left(\left\{\frac{C}{I} > y\right\}\right) \overset{(a)}{=} \int_{r_1=0}^{\infty} \mathbb{P}\left(\frac{C}{I} > y \middle| R_1 = r\right) f_{R_1}(r)dr \overset{(b)}{=} \int_{q=0}^{\infty} \mathbb{P}\left(\frac{C}{I} > y \middle| Q = q\right) f_Q(q)dq$, where $Q \triangleq \mu(R_1)$, and Q is an exponential random variable with mean 1.

Equation (a) is obtained by conditioning w.r.t. R_1. Equation (b) is obtained by expressing (a) in terms of Q, where the p.d.f. of R_1 at $R_1 = \mu^{-1}(q)$ is $f_{R_1}(r)dr|_{r=\mu^{-1}(q)} = e^{-\int_0^r \lambda(s)ds}\lambda(r)dr\Big|_{r=\mu^{-1}(q)} = e^{-q}dq = f_Q(q)dq$, which does not depend on $\lambda(r)$.

To show that the BS density $\lambda_1(r)$ gives a worse $\frac{C}{I}$ than $\lambda_2(r)$ does, one needs to show that $\frac{C}{I}\Big|_{R_1=\mu_1^{-1}(q),\ \lambda_1(r),\ r\geq\mu_1^{-1}(q)} \leq_{\mathrm{st}} \frac{C}{I}\Big|_{R_1=\mu_2^{-1}(q),\ \lambda_2(r),\ r\geq\mu_2^{-1}(q)}$ for all $q > 0$, where the condition of the domain of the BS density is because the locations of interfering BSs only depend on the BS density in that domain. Next, define $a = \frac{\mu_2^{-1}(q)}{\mu_1^{-1}(q)}$. By Corollary 1, where R_k''s are the ordered BS locations of the SCS with BS density $\frac{1}{a}\lambda\left(\frac{r}{a}\right)$. The equation means that the conditional $\frac{C}{I}$ of the SCS with a BS density $\lambda_1(r)$ is equivalent to an SCS with BS density $\frac{1}{a}\lambda_1(\frac{r}{a})$ with the same location of the serving BS as the SCS with BS density $\lambda_2(r)$.

With the locations of the serving BSs equal and fixed, $\frac{C}{I}$ is a decreasing function of the interference. Theorem 1.A.3.(a) of [7] says that decreasing functions reverse the usual stochastic order. So, one only needs to show that the interferences satisfy

$$\sum_{k=2}^{\infty} R_k^{-\varepsilon}|_{R_2\geq\mu_2^{-1}(q),\ \frac{1}{a}\lambda_1(\frac{r}{a}),\ r\geq\mu_2^{-1}(q)} \geq_{\mathrm{st}} \sum_{k=2}^{\infty} R_k^{-\varepsilon}|_{R_2\geq\mu_2^{-1}(q),\ \lambda_2(r),\ r\geq\mu_2^{-1}(q)}. \quad (1)$$

As shown in [2, Appendix B], the total interference power can be expressed as $\sum_{k=2}^{\infty} R_k^{-\varepsilon} = \lim_{r_B\to\infty}\lim_{N\to\infty}\sum_{i=2}^{N} X_i$, where X_i is a Bernoulli random variable defined by $\mathbb{P}\left(\{X_i = 0 | R_1 = r_1\}\right) = 1 - p_i$, $\mathbb{P}\left(\left\{X_i = r_i^{-\varepsilon} + o(\Delta r) \middle| R_1 = r_1\right\}\right) = p_i$, $p_i = \lambda(r_i)\Delta r + o(\Delta r)$, $r_i = r_1 + (i-1)\Delta r$, $\Delta r = \frac{r_B - r_1}{N}$, and $r_1 = \mu_2^{-1}(q)$. Since the condition $\frac{1}{a}\lambda_1(\frac{r}{a}) \geq \lambda_2(r)$ holds for all $r \geq \mu_2^{-1}(q)$, we have $X_i|_{\frac{1}{a}\lambda_1(\frac{r}{a})} \geq_{\mathrm{st}} X_i|_{\lambda_2(r)},\ \forall i \geq 2$. As summation preserves stochastic order [7, Theorem 1.A.3.(b)], (1) is proved, completing the proof.

References

[1] T. X. Brown, "Cellular performance bounds via shotgun cellular systems," *IEEE J. Sel. Areas Commun*, vol. 18, no. 11, pp. 2443–2455, Nov. 2000.

[2] P. Madhusudhanan, J. G. Restrepo, Y. Liu, T. X. Brown, and K. Baker, "Generalized carrier to interference ratio analysis for the shotgun cellular system," *CoRR*, 2010. Available: http://arxiv.org/abs/1002.3943

[3] ——, "Modeling of interference from cooperative cognitive radios for low power primary users," in *Proc. 2010 IEEE Globecom Wireless Communications Symposium*, pp. 1–6.

[4] ——, "Multi-tier network performance analysis using a shotgun cellular system," in *Proc. 2011 IEEE Globecom Wireless Communications Symposium*, pp. 1 –6.

[5] C. Tepedelenlioglu, A. Rajan, and Y. Zhang, "Applications of stochastic ordering to wireless communications," *IEEE Trans. Wireless Commun.*, vol. 10, no. 12, pp. 4249–4257, Dec. 2011.

[6] B. Błaszczyszyn and D. Yogeshwaran, "Directionally convex ordering of random measures, shot-noise fields and some applications to wireless networks," *Adv. Appl. Probab.*, vol. 41, pp. 623–646, 2009.

[7] M. Shaked and J. Shanthikumar, *Stochastic Orders*, ser. Springer Series in Statistics. Springer, 2007.

[8] J. F. C. Kingman, *Poisson Processes (Oxford Studies in Probability)*. Oxford University Press, 1993.

[9] S. M. Ross, *Stochastic Processes*. John Wiley & Sons, Inc., 1983.

[10] T. S. Rappaport, *Wireless Communications: Principles and Practice*. Prentice-Hall, Inc., 1996.