A REDUCED DESCRIPTION OF RAPIDLY ROTATING TURBULENT CONVECTION

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1. Introduction

The tendency towards two-dimensionality in rapidly rotating flows is described by the Taylor-Proudman theorem. In applications strict two-dimensionality is usually broken by the presence of boundary and/or thermal forcing, or by initial conditions. Both types of forcing are present in thermal convection in a rapidly rotating horizontal layer, described by the equations

$$\frac{D\mathbf{u}}{Dt} + \hat{\mathbf{\Omega}} \times \mathbf{u} = -\nabla\pi + E\nabla^2\mathbf{u} + \frac{Ra}{\sigma}E^2T\hat{\mathbf{z}}, \qquad \nabla\cdot\mathbf{u} = 0. \tag{1}$$

$$\sigma\frac{DT}{Dt} = E\nabla^2 T. \tag{2}$$

Here $E \equiv \nu/2\Omega d^2$ is the Ekman number, assumed to be small, and Ra and σ are the Rayleigh and Prandtl numbers; distances have been expressed in units of the layer depth d and time in units of the Coriolis time $(2\Omega)^{-1}$. All other symbols have their usual meaning. In the following we employ a multiple scale expansion in both time and space, and focus on horizontal scales on which the horizontal and vertical velocities are comparable, but the horizontal velocity components are still in geostrophic balance.

2. Reduced Interior Equations

We introduce "fast" horizontal variables $x' \equiv E^{-\frac{1}{3}}x$, $y' \equiv E^{-\frac{1}{3}}y$ and use the notation $D \equiv \partial_z$ to denote derivatives with respect to the "slow" variable z. In thermal convection these are the dominant scales selected by linear theory. We also introduce a slow time $t' \equiv E^{\frac{1}{3}}t$. Finally, we write $\mathbf{u} = \nabla \times \phi\hat{\mathbf{z}} + \nabla \times \nabla \times \psi\hat{\mathbf{z}}$ and scale the streamfunctions according to $\phi = E\phi'$, $\psi = E^{\frac{4}{3}}\psi'$. This rescaling implies that on the length scales of interest the vertical and horizontal velocities are of the same order, both $\mathcal{O}(E^{\frac{2}{3}})\Omega d$ in dimensional units. The local Rossby number is thus $\mathcal{O}(E^{\frac{1}{3}})$ and hence is small; i.e., even though the scales of interest are small, the flow is still rotation dominated. Moreover, with this scaling the horizontal velocity components are in geostrophic balance at leading order. Finally, we introduce the scaled Rayleigh number $Ra' \equiv E^{\frac{4}{3}}Ra$ and split the temperature T into its mean and fluctuating parts: $T(x,y,z,t) \equiv \overline{T}(z) + E^{\frac{1}{3}}\theta(x,y,z,t)$. Here, the overbar denotes a spatial average in the horizontal *and* a time-average. The time-averaging is an essential aspect of our decomposition, and allows us to close the problem (Julien and Knobloch 1998).

On dropping all primes we obtain the following reduced system,

$$\partial_t \nabla_\perp^2 \phi - J[\phi, \nabla_\perp^2 \phi] - D\nabla_\perp^2 \psi = \nabla_\perp^4 \phi + \mathcal{O}(E^{\frac{1}{3}}) \ , \tag{3}$$

$$\partial_t \nabla_\perp^2 \psi - J[\phi, \nabla_\perp^2 \psi] + D\phi = -\frac{Ra}{\sigma}\theta + \nabla_\perp^4 \psi + \mathcal{O}(E^{\frac{1}{3}}) \ , \tag{4}$$

i.e., a pair of equations for the vertical velocity $w \equiv -\nabla_\perp^2 \psi$ and the vertical vorticity $\zeta \equiv -\nabla_\perp^2 \phi$, coupled via vertical stretching and driven by thermal buoyancy:

$$\sigma(\partial_t \theta - J[\phi, \theta] - \nabla_\perp^2 \psi D\overline{T}) = \nabla_\perp^2 \theta + \mathcal{O}(E^{\frac{1}{3}}) \ . \tag{5}$$

Here $J[f,g] \equiv \partial_x f \partial_y g - \partial_y f \partial_x g$ is the horizontal Jacobian operator. The equations are closed using the first integral of the mean temperature equation:

$$D\overline{T} = -1 - \sigma\overline{(\nabla_\perp^2 \psi\, \theta - \langle \nabla_\perp^2 \psi\, \theta \rangle)} + \mathcal{O}(E^{\frac{1}{3}}) \ . \tag{6}$$

Here the angular brackets denote a vertical average. The resulting equations describe the dynamics in the bulk, outside of any Ekman boundary layers required by horizontal boundaries. In the present problem such boundaries are passive (Julien and Knobloch 1998).

3. Results

Equations (3–6) were solved numerically using a pseudo-spectral Petrov-Galerkin method in which field variables are represented with sines or

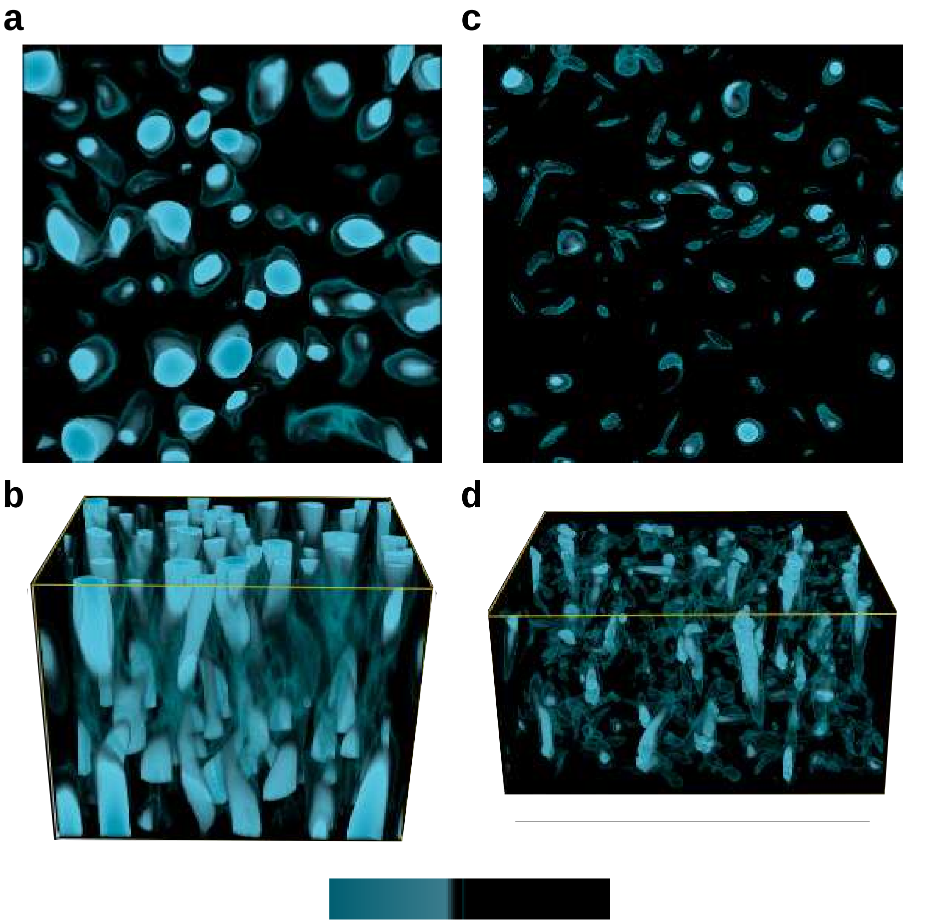

Figure 1. Figure 1. Results of direct numerical simulations of the reduced equations (a,b) and the full three-dimensional Boussinesq equations (c,d). Top views are depicted in (a) and (c), while perspective views are shown in (b) and (d). The visualization of vortical structures is achieved with the quantity λ_2, the intermediate eigenvalue of R^2+S^2, where R and S are the rotation and strain matrices of the velocity field. The negative values shown in grey (see color bar) indicate vortex tubes (Jeong and Hussain, 1995). The reduced equations are simulated with $Ra = 20E^{-\frac{4}{3}}$ and an aspect ratio of $6E^{-\frac{1}{3}} \times 6E^{-\frac{1}{3}} \times 1$ (not to scale); they capture the coherent structures of the full Boussinesq equations (first reported in Julien et al 1996). The parameter values are $Ra = 1.0 \times 10^7$, $E = 9.4 \times 10^{-5}$, and an aspect ratio of $2 \times 2 \times 1$ (to scale); the effective Rossby number is 0.3. The aspect ratio selection for the reduced problem was based on the number of critical modes (as determined by linear theory) that fit into the full three-dimensional Boussinesq simulation.

cosines in the vertical and periodic Fourier modes in the horizontal. Time-stepping is via a mixed implicit/explicit 3rd-order Runge-Kutta scheme developed by Spalart et al (1991), with the diffusion and forcing terms treated implicitly and nonlinear and stretching terms explicitly. We set $Ra' = 20$ (2.2 times critical), $\sigma = 1$, and the domain width to 6 times the most unstable linear wavelength ($\lambda_\perp \approx 4.8154$) in both horizontal directions. Top and bottom boundary conditions are impenetrable/fixed temperature and side boundaries are periodic. The calculations were conducted with 64^3 spectral modes and were de-aliased in all spatial directions at each Runge-Kutta sub-timestep.

Fig. 1a–b shows a sample solution obtained using only spatial averaging for the mean terms: in a sufficiently large domain (as in Fig. 1a–b) the horizontal average of the rising and falling plumes becomes equivalent to a time-average. After a very short initial transient, vortical buoyant plumes emerge and mutually advect one another laterally. The plumes are columnar, spanning the layer depth, as one expects from the Taylor-Proudman constraint. Very near the boundaries, however, sharp gradients appear, as anticipated from the thermal boundary conditions. These features resemble closely those found in numerical solutions of the primitive equations (1,2) at large rotation rates (Fig. 1a–b). A detailed comparison of the two sets of solutions will appear elsewhere.

Equations (3,4) with $Ra = 0$ describe decaying rapidly rotating turbulence and merit study in their own right (Julien et al 1998).

Acknowledgment: This research was initiated while EK was a Visiting Fellow at JILA, University of Colorado, Boulder, and partially supported by the Department of Energy under grant DE-FG03-95ER-25251. JW is partially supported by an NSF High Performance Computing and Communication Grand Challenge grant in geophysical and astrophysical fluid turbulence (ECS9217394). The numerical calculations were conducted at the Pittsburgh Supercomputing Center under MetaCenter grant MCA935010P.

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