Problem 1 (How useful is the Cholesky decomposition?)

Perform a timing study in R to compare the savings of using the Cholesky decomposition to calculate the pieces of a multivariate normal likelihood.

For sample sizes \( n = 100, 500, 1000, 1500, 2000, 2500, 3000 \) calculate the time it takes to compute the determinant and quadratic form for a mean zero Gaussian process having exponential covariance with range of 1 with samples on the interval \([0,10]\). Use the \texttt{system.time} function in R to calculate times. Do so for comparing the Cholesky approach to the “lazy” approaches:

(a) \( \text{det}(\Sigma) \) via

\[
\text{determinant(Sigma,log=TRUE)}$\mod vs. 2*sum(log(diag(chol(Sigma))))
\]

(b) \( y^T\Sigma^{-1}y \) via

\[
t(y)^*\%solve(Sigma)^*%y vs. sum(forwardsolve(t(chol(Sigma)),y)^2).
\]

For each part, plot both timings using the lazy method vs. the Cholesky method, in seconds, as a function of sample size on the same graph.

Problem 2 (The difference between range and smoothness)

Simulate a 1-dimensional mean zero Gaussian process with a Matérn correlation having range of \( a = 0.2 \) and a smoothness of \( \nu = 1.5 \). Do so for

(a) 50 equally spaced points on \([0,10]\]
(b) 50 equally spaced points on \([0,1]\).

For both simulations, find the MLEs for \( a \) and \( \nu \), and find 95% confidence intervals based on the estimated (inverse) Fisher information matrix. Comment on which case seems to produce better estimates for each parameter. Given the interpretations of these parameters, do your findings make sense?
Problem 3 (OLS vs. ML/GLS)

In Homework 3, you modeled MODIS satellite estimates of sea surface temperature over an ocean. Denote the $T$ variable at spatial location $s$ by $T(s)$. Most of you considered a model of the form

$$T(s) = \beta_0 + \beta_1 \text{lat}(s) + \beta_2 \text{lon}(s) + Z(s) + \varepsilon(s)$$

where the mean function includes linear variation over latitude and longitude. In this problem, suppose $Z(s)$ is a mean zero Gaussian process with an exponential covariance function ($C(h) = \sigma^2 e^{-\|h\|/a}$) and $\varepsilon(s)$ is a mean zero nugget effect with variance $\tau^2$.

In this problem, compare estimates for $\beta = (\beta_0, \beta_1, \beta_2)$ and $\theta = (a, \sigma^2, \tau^2)$ where:

(a) $\beta$ is estimated by ordinary least squares and $\theta$ is estimated based on the residuals by fitting a binned empirical semivariogram.

(b) $\theta$ is estimated by maximum likelihood using the profile log-likelihood, and $\beta$ is the generalized least squares estimator.

Are your estimates the same, and if not, how do they differ?

Problem 4 (Grad students only)

If $\Sigma$ is the covariance matrix for a random vector $Z$, and $a$ and $b$ are orthogonal vectors, are $a^T Z$ and $b^T Z$ uncorrelated random variables?

(a) Find an example where $a^T Z$ and $b^T Z$ are correlated.

(b) Find sufficient conditions on $a$ and $b$ that ensure $a^T Z$ and $b^T Z$ are uncorrelated (hint: use the spectral/eigen decomposition of $\Sigma$; what if $a$ and $b$ are two different eigenvectors of $\Sigma$?).