Problem 1 (Variance-covariance matrices of OLS/GLS estimators)

Consider a linear model of the form
\[ Y = X\beta + \varepsilon \]
where \( \text{Var} \varepsilon = \Sigma \) (and so \( \text{Var} Y = \Sigma \)). Let
\[ \beta_{OLS} = (X^TX)^{-1}X^TY \]
and
\[ \beta_{GLS} = (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1}Y \]
be the ordinary least squares and generalized least squares estimators for \( \beta \), respectively. Find

(a) \( \text{Var}\beta_{OLS} \).
(b) \( \text{Var}\beta_{GLS} \).
(c) Specialize your result from part (a) for the case where \( \beta = \beta_0 \) is a single mean parameter and \( \text{Var} Y = (C(||s_i - s_j||))_{i,j=1}^n \), i.e., the errors are spatially correlated with isotropic correlation function \( C \).

Problem 2 (Simulation)

Suppose \( Z(s) \) is a stochastic process on \( s \in \mathbb{R} \) defined by
\[ Z(s) = W_0 + W_1s + W_2s^2 + W_3s^3 \]
where \( W_0, W_1, W_2 \) and \( W_3 \) are iid standard normal random variables.

(a) Simulate this stochastic process on a grid of 1000 points covering \([0, 1]\) five times and plot your simulations on the same plot using different colors.
(b) Repeat part (a) when the coefficient variables are iid \( U(0, 1) \).
Problem 3 (Empirical covariances)

If $Z(s)$ is a process observed at spatial locations $s_1, \ldots, s_n$, then one version of an empirical covariance function is defined as

$$
\hat{C}(r) = \frac{1}{|N(r)|} \sum_{(s_i, s_j) \in N(r)} (Z(s_i) - \overline{Z})(Z(s_j) - \overline{Z})
$$

where $N(r)$ is a set of pairs of locations whose distance is approximately $r$ and $|N(r)|$ is the number of pairs in that set and $\overline{Z}$ is the simple arithmetic average over all data.

This problem uses the data file `ThreeSurfaces2.RData` on the class website. In the workspace there are three $80 \times 80$ surfaces, `dat1`, `dat2` and `dat3`. Spatial locations for each plot are on a grid $\{(i,j)\}_{i,j=1}^{80}$.

(a) Plot all three surfaces in a row using `image.plot` in the `fields` package, and comment on the visual differences.

(b) Using the `vgram` function in `fields`, calculate and plot $\hat{C}(r)$ for values of $0 \leq r \leq 50$ using the same number of bins for each dataset. Comment on the differences as compared to your plots from part (a). Do these align with your expectations based on part (a)? Are the differences striking?

Problem 4 (Grad students only)

Using the stochastic process from Problem 2(a), find the joint distribution of $(Z(0), Z(s))^T$ for $s > 0$, and comment on the dependence structure: what happens to $Z(s)$ as $s \to \infty$ and what happens to the correlation between $Z(0)$ and $Z(s)$?