MECHANICS OF MATERIALS WITH PERIODIC TRUSS OR FRAME MICRO-STRUCTURES

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ABSTRACT. This paper describes the mechanics of materials with periodic skeletal micro-structures. The principal technical results consist of certain Korn-type inequalities for solutions to the elasto-static equilibrium equations; these inequalities provide upper and lower bounds for the linear elastic strain energy in the material. Using these bounds, existence and uniqueness results for the equations of linear elastic equilibrium are derived, and certain asymptotic properties of the solutions are described. Particular attention is paid to the question of when a lattice structure can accurately be modelled as a pin-jointed truss, and when a rigid-node frame model must be employed. A practical technique for how to distinguish between the two types of material is given, and the distinct differences in their mechanical behavior are described.

1. INTRODUCTION

1.1. **Motivation.** In this paper we describe the mechanics of materials that have a periodic skeletal micro-structure, called *lattice materials*. It has long been known that through design of the micro-structural geometry, it is possible to engineer such materials that have extreme or unusual properties but it is only during the last decade that manufacturing technology has advanced sufficiently to allow industrial scale production of materials of this kind. This development has prompted a surge of interest from investigators who have proposed the design of, *e.g.* (i) materials that block acoustic waves in certain frequency bands, Sigmund and Jensen [29], Ruzzene and Scarpa [26], and Martinsson and Movchan [20], (ii) materials that have negative Poisson's ratios or negative thermal expansion coefficients, Lakes [14] and Sigmund and Torquato [30], (iii) materials with very low stiffness-to-weight ratios, Gibson and Ashby [13], (iv) materials with very high stiffness-to-weight ratios, Deshpande, Fleck and Ashby [7] and Wallach and Gibson [33], and, (v) materials where the nature of the micro-structure is used to construct multi-functional materials, Evans, Hutchinson and Ashby [9].

The motivation for the research presented in this paper is twofold: (1) To obtain the detailed quantitative description of the lattice equilibrium equation that is necessary for the construction of fast numerical solvers, see *e.g.* [18]. (2) To answer basic questions regarding the connection between the micro-structural geometry of a lattice material and its macro-scopic properties. As a simple illustration of such a question, consider the case of in-plane loads acting on the lattice materials in Fig. 2.1. It is well known that structures B and C will be much stiffer than structures A and D since any deformation of B or C would require struts to change

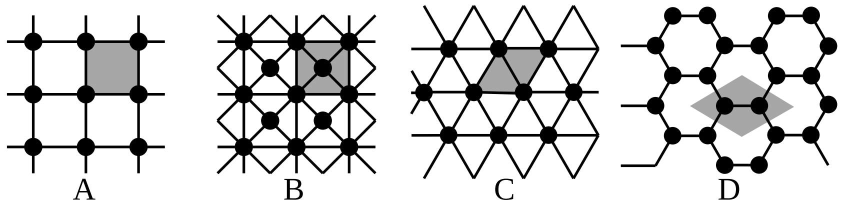

FIGURE 1.1. Examples of lattices. An irreducible unit cell is shaded. Lattices A and C are mono-atomic, B and D are multi-atomic. Lattices B and C are of truss type, A and D of frame type.

length, thus engaging their axial stiffness rather than their bending stiffness, see Gibson and Ashby [13]. We call stiff materials such as B and C *truss*-materials and soft materials such as A and D *frame*-materials. We will answer the question of how to determine to which of these two classes a given geometry belongs.

1.2. **Scope of the present paper and summary of main results.** We study the equations associated with the following three lattice models:

- *Thermostatics of an infinite lattice:* The equilibrium equation relates an unknown field of nodal temperatures to a prescribed field of nodal heat sources.
- *Elastostatics of an infinite truss:* The struts are modelled as axial springs that are connected by pin-joints at the nodes. The equilibrium equation relates a field of nodal displacements to a field of prescribed forces. In $\mathbb{R}^d$, the nodal potential has dimension d.
- *Elastostatics of an infinite frame:* The struts are modelled as beams with both axial and bending stiffnesses. The beams are rigidly connected at the nodes. The equilibrium equation relates a field of translational and rotational nodal displacements to a prescribed field of forces and torques. In $\mathbb{R}^d$, the nodal potential has dimension $d(d+1)/2$.

The equilibrium equations under consideration are defined on the integer lattice $\mathbb{Z}^d$ and are elliptic in nature. For the thermostatics problem, the lattice equation is qualitatively similar to Poisson's equation. For the elastostatics problem on truss and frame lattices, the analogous continuum equations are the equations of classical and micro-polar elasticity, respectively.

We consider only infinite periodic lattices but make very weak assumptions on the local geometry. In fact, we prove that for heat conduction and the mechanical frame model, the condition that the lattice is connected is both sufficient and necessary for the equilibrium equation to be well-posed. For the mechanical truss model, the corresponding condition is slightly more complex to state (although it is simple to use in practise). We prove that the given condition is sufficient for well-posedness of the equilibrium equation, and we conjecture that it is necessary, although this has not yet been proved.

Remark 1.1. Lattice equations on *finite* domains can be analyzed using a discrete boundary formulation, see Saltzer [27, 28] and [18, 25]. Such a formulation relies on the existence of a fundamental solution to the lattice equilibrium equation, which is assured due to the Korn-type inequalities proved in this paper, see [21].

For each model mentioned above, we prove that the lattice equilibrium equation is coercive and satisfies a Korn-type inequality. These inequalities are used to prove that solutions exist and are unique (up to rigid body motions) as long as the load satisfies modest decay conditions. Under slightly stronger conditions on the right hand side, we construct an explicit inverse operator using Fourier techniques and describe its basic properties. For mono-atomic lattices, the results presented are closely analogous to the corresponding continuum theories (heat conduction, classical elasticity for truss problems and micro-polar elasticity for frame problems). For multi-atomic lattices, additional effects such as solutions with intra-cell oscillations are present.

Remark 1.2. The truss and the frame models are related as follows: The frame model accurately captures the mechanics of any lattice material with slender struts. However, this is a very complex model and for many lattice geometries, such as B and C in Figure 1.1, the bending stiffnesses of the struts contribute only marginally to the overall strength of the material. To a high degree of accuracy, they can simply be ignored, which then leads to the truss model. Strict bounds on the modelling error thus incurred can be constructed using a technique outlined in Section 7.5.

1.3. **Mathematical apparatus.** In this section we first give a brief review of one way of using Fourier methods to study the continuum equations of elasticity. We then describe how these methods can be modified to study the discrete equations that describe elastic equilibrium on a lattice.

Consider static equilibrium of an infinite elastic body. Using standard tensor notation, we let u_i denote the (unknown) displacement field, f_i the (prescribed) body forces, C_{ijkl} the constant stiffness tensor and $x = [x_1, \dots, x_d]$ a point in $\mathbb{R}^d$. The equilibrium equation then reads

$$-\sum_{i,k,l=1}^{d} C_{ijkl} \frac{\partial^2}{\partial x_k \partial x_l} u_i(x) = f_j(x), \qquad \forall\, x \in \mathbb{R}^d. \tag{1.1}$$

For $\xi = [\xi_1, \dots, \xi_d] \in \mathbb{R}^d$, we introduce a Fourier transform

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{\mathbf{i}x\cdot\xi} u(x)\, dx, \tag{1.2}$$

and apply it to the equation (1.1) to obtain the alternative form

$$\sum_{i,k,l=1}^{d} C_{ijkl}\, \xi_k\, \xi_l\, \hat{u}_i(\xi) = \hat{f}_j(\xi). \tag{1.3}$$

We then rewrite (1.3) as

$$\sigma(\xi)\, \hat{u}(\xi) = \hat{f}(\xi), \qquad \forall\, \xi \in \mathbb{R}^d, \tag{1.4}$$

by introducing the symbol $\sigma(\xi)$ of the equilibrium operator. It is a $d \times d$ matrix whose ij'th component is given by

$$[\sigma(\xi)]_{ij} := \sum_{k,l=1}^{d} C_{ijkl}\, \xi_k\, \xi_l. \tag{1.5}$$

Since the equation (1.4) is algebraic, it can trivially be solved; $\hat{u}(\xi) = \sigma(\xi)^{-1}\hat{f}(\xi)$. Upon application of the Fourier inversion formula, we formally find that

$$u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\mathbf{i}x\cdot\xi} \sigma(\xi)^{-1} \hat{f}(\xi)\, d\xi. \tag{1.6}$$

The questions of existence, uniqueness and stability of solutions to (1.4), and well-posedness of the integral (1.6), all hinge on the identity

$$\overline{\varphi} \cdot \sigma(\xi)\varphi \geq c|\xi|^2|\varphi|^2, \qquad \forall\, \varphi \in \mathbb{C}^d, \tag{1.7}$$

which is a crude incarnation of Korn's inequality.

When studying lattice materials, we employ a structural mechanics model so that the unknown displacement field is defined by its nodal values. Considering a truss-lattice with q nodes in the unit cell, we use $m \in \mathbb{Z}^d$ as an index for a cell and let $\boldsymbol{u}(m) \in \mathbb{R}^{qd}$ denote the displacements of the q nodes in the cell. The equilibrium equation then takes the form, *cf.* (1.1),

$$[\boldsymbol{A}\boldsymbol{u}](m) = \boldsymbol{f}(m), \qquad \forall\, m \in \mathbb{Z}^d, \tag{1.8}$$

where $\boldsymbol{f}$ is the prescribed load and the operator $\boldsymbol{A}$ is a constant coefficient convolution operator

$$[\boldsymbol{A}\boldsymbol{u}](m) = \sum_{n\in\mathbb{B}} A^{(n)}\boldsymbol{u}(m-n).$$

Here, $\mathbb{B}$ is a *finite* index set and the $A^{(n)}$'s are matrices. (We use boldface italic symbols to define functions on lattices and operators acting on such functions.) Introducing a discrete Fourier transform

$$\tilde{\boldsymbol{u}}(\xi) = [\boldsymbol{F}\boldsymbol{u}](\xi) = \sum_{m\in\mathbb{Z}^d} e^{\mathbf{i}m\cdot\xi}\boldsymbol{u}(m), \qquad \forall\, \xi \in (-\pi,\, \pi)^d =: I^d, \tag{1.9}$$

and applying it to both sides of (1.8) we obtain the diagonal form, *cf.* (1.4),

$$\sigma(\xi)\tilde{\boldsymbol{u}}(\xi) = \tilde{\boldsymbol{f}}(\xi), \qquad \forall\, \xi \in I^d, \tag{1.10}$$

where the symbol of $\boldsymbol{A}$ is given by

$$\sigma(\xi) = \sum_{n\in\mathbb{B}} e^{\mathbf{i}n\cdot\xi} A^{(n)}. \tag{1.11}$$

Solving (1.10) and applying the inverse of (1.9) we find that

$$\boldsymbol{u}(m) = \frac{1}{(2\pi)^d} \int_{I^d} e^{-\mathbf{i}n\cdot\xi} \sigma(\xi)^{-1} \tilde{\boldsymbol{f}}(\xi)\, d\xi. \tag{1.12}$$

In order to state the lattice equivalent of (1.7) it is necessary to split the displacements associated with a cell into a cell-wise average and an intra-cell oscillation. On the Fourier domain, this amounts to splitting a vector $\varphi \in \mathbb{C}^{qd}$ into an average

φ_{a} and a difference φ_{d} so that $\varphi = \varphi_{\mathrm{a}} + \varphi_{\mathrm{d}}$. Thus, φ_{a} belongs to a d-dimensional subspace of $\mathbb{C}^{qd}$. Then there exists a $c > 0$ such that

$$\overline{\varphi} \cdot \sigma(\xi)\varphi \geq c\left(|\xi|^2|\varphi_{\mathrm{a}}|^2 + |\varphi_{\mathrm{d}}|^2\right), \qquad \text{for all } \varphi \in \mathbb{C}^{qd}. \tag{1.13}$$

The proof that (1.13), and the analogous statement for frame lattices, hold is the main contribution of the present work. Once these inequalities are established, it is straight-forward to prove that (1.8) is well-posed and that the integral (1.12) is well-defined under appropriate conditions on $\boldsymbol{f}$.

In proving the coercivity result (1.13), three technical obstacles must be overcome: (i) The mechanics of the average fields and the intra-cell oscillations must be separated. (ii) A non-degeneracy condition that excludes lattice geometries that can deform without any bars changing length (such as A and D in Figure 1.1) must be devised and incorporated into the analysis of truss lattices. (iii) In the analysis of frame lattices, the rotational degrees of freedom must be treated differently than the translational ones.

Remark 1.3. The inequality (1.13) admits a detailed description of the singular behavior of $\sigma(\xi)^{-1}$ as $\xi \to 0$. Recalling that small values of ξ in Fourier space correspond to long range modes in physical space (where "long" is measured against the cell size) we expect that the nature of this singularity determines the asymptotic behavior of solutions to the equilibrium equation (1.8) as the lattice cell size tends to zero. In fact, once the singular behavior of $\sigma(\xi)^{-1}$ is known, it is possible to construct continuum equations (so called "homogenized" equations) whose solutions approximate the solution of the lattice equilibrium equation to arbitrary order of accuracy. See [19] for details.

Remark 1.4. Equations of the form (1.8) also occur in the modelling of atomic crystals and many other applications. The results presented are readily generalizable to such models.

1.4. **Context.** The problem of finding the energy minimizer for a network problem can be approached from several different directions. In the current work, we have decided to limit the analysis to the case of quadratic energy functionals (in other words, the "small deformations" case) defined on periodic, infinite networks. This allows us to make truly minimal assumptions on the geometry of the lattice. (The generality is important since complicated multi-atomic lattices are frequently encountered in applications.) Moreover, in this environment it was possible to extend the analysis of the conduction problem (which has been studied by several authors, see, *e.g.* Vogelius [32]) to the case of mechanical lattices whose links have both bending and torsional stiffnesses.

In contrast, the case of non-quadratic energy functionals, generally with more restrictive geometric assumptions, has been studied by, *e.g.* Blanc, LeBris and Lions [4], Braides and Gelli [5], and Connelly and Whiteley [6].

The question of when an infinite periodic truss structure is rigid has previously been addressed by Babuška and Sauter [2]. The analysis in [2] is based on algebraic methods and the test criterion is very different from the one presented here. The relationship between the two criteria is currently under investigation. There is also an extensive litterature on the rigidity of finite truss structures, see *e.g.* Whiteley [34] and Laman [16]. A review of the engineering literature on this subject is provided by Ostoja-Starzewski [23]. A description of Green's function techniques for analyzing finite lattice structures is given in Liu, Karpov and Park [17].

For methods applicable to non-periodic network problems, we refer to Berlyand and Kolpakov [3], and the references therein.

As mentioned in Remark 1.1, a principal motivation for the current work is to enable the construction of a fundamental solution to the equilibrium equation on an infinite lattice. Such a fundamental solution can then be used to develop analytical and computational techniques for treating the case of equilibrium problems on finite lattice structures by rewriting the equilibrium equation on the domain as an equation on the boundary, see [25]. However, since the analysis is limited to the case of quadratic energy functionals, such an approach cannot be used to treat the problem of finding large deformation energy minimizers for problems on finite structures. An illustration of how delicate such problems can be even on very simple lattice structures is given by Friesecke and Theil [12].

From a technical point of view, we were much helped by the earlier literature on the use Fourier transforms to study difference equations, see *e.g.* Duffin [8], Fix and Strang [10, 11], Stephan [31], and Babuška [1]. In the early stages of the research, invaluable insight into the mechanics of the truss- and frame-materials was obtained from the rich engineering literature on the subject. In addition to the works mentioned earlier in this section, we particularly wish to mention Lakes [15] and Noor [22] who discuss the use of micro-polar models of elasticity for modelling bending dominated structures.

1.5. **Outline.** This paper is organized as follows: In Section 2, we introduce notation for describing lattice geometries and lattice potentials. We also describe some known facts about the discrete Fourier transform and Hermitian matrices. In Section 3, we derive the lattice equilibrium equation in a general setting that covers, in particular, the three models that we study. We also prove that the equilibrium operator is bounded on l^2, self-adjoint and positive semi-definite. In Section 4, we leave the general case to study the conduction model. We prove the Korn-type inequality (1.13) and use it to demonstrate that after factoring out constant functions, the equilibrium operator is positive definite. This is then used to prove that the lattice equilibrium equation is well-posed and to describe the nature of the inverse. In Section 5 we extend some of the results for the scalar case to general lattice models. In sections 6 and 7, these results are applied to investigate the truss and frame models, respectively. Section 8 summarizes the main results.

2. Preliminaries

In this section, we establish a framework for describing lattice geometries and functions defined at the nodes of a lattice. We will also introduce a discrete Fourier transform and present some basic facts about Hermitian positive definite matrices.

For simplicity, we restrict attention to lattices with the unit cube $[0,\, 1)^d$ as the irreducible cell. This is for notational convenience only, other geometries (such as

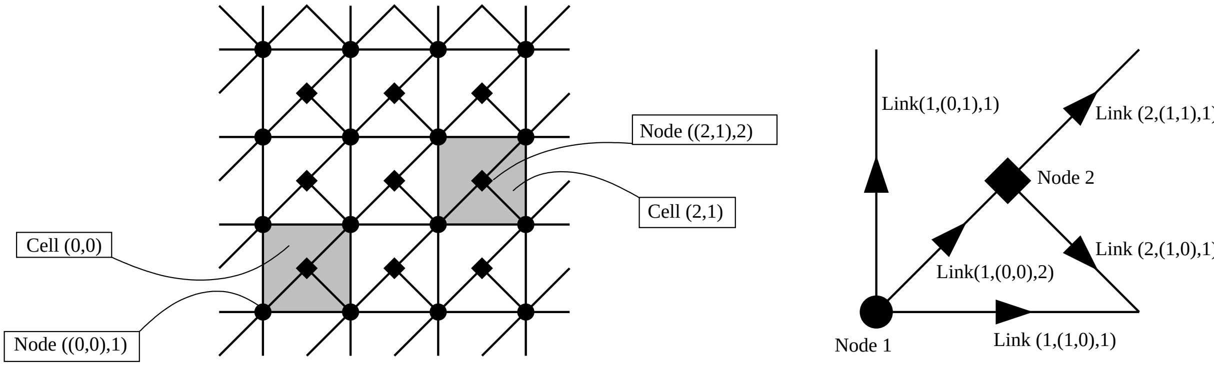

FIGURE 2.1. Lattice notation. Circles denote nodes of type 1 and diamonds nodes of type 2. The unit cell is depicted on the right.

C and D in Figure 1.1) can be treated by simply introducing a scaling matrix into all formulas, see [18].

2.1. **General notation.** Given a matrix A, we let A^{t} denote its transpose and A^* its adjoint (the complex conjugate transpose). For a vector v, we let $|v|_p$ denote the l^p-norm. Given a Hermitian positive semi-definite matrix X, we define

$$\langle \varphi, X, \psi \rangle = \overline{\varphi} \cdot X\psi = \overline{X\varphi} \cdot \psi.$$

Furthermore, if X and Y are both Hermitian and $\langle \varphi, X, \varphi \rangle \leq \langle \varphi, Y, \varphi \rangle$ for every φ, then we say that $X \leq Y$. If there are positive constants a and b, such that $aX \leq Y \leq bX$, then we say that $X \sim Y$. The following is a well-known result but we state it for future reference.

Lemma 2.1. *If X and Y are Hermitian positive semi-definite matrices and $X \leq Y$, then $\det X \leq \det Y$. If X and Y are also invertible, then $Y^{-1} \leq X^{-1}$.*

2.2. **Describing the lattice geometry.** For a d-dimensional lattice with q nodes in the unit cell, we use $m \in \mathbb{Z}^d$ to label the different cells and $\kappa \in \{1, \dots, q\} = \mathbb{N}_q$ to label the different nodes in a cell. Thus, (m, κ) uniquely labels a node in the lattice $\Omega := \mathbb{Z}^d \times \mathbb{N}_q$. Let $X(m, \kappa) \in \mathbb{R}^d$ denote the coordinates of node (m, κ). The periodicity of the lattice then implies that $X(m, \kappa) = X(0, \kappa) + m$. Finally, define for non-negative integers J, the sub-lattices $\Omega_J := \{(m, \kappa) \in \Omega \,:\, |m|_\infty \leq J\}$.

In addition to the points in Ω, the lattice also consists of directed links that connect these points. Since the lattice is periodic, it is sufficient to describe only those links that originate in the zero-cell, Ω_0. A link from the node $(0, \kappa)$ to the node (n, λ) is then represented by the triple $(\kappa, n, \lambda) \in \mathbb{N}_q \times \mathbb{Z}^d \times \mathbb{N}_q$. All such triples are collected in a set $\mathbb{B}_+$. In applications where the links are innately direction-less (as they are in the conduction and truss models but not in the frame model) we arbitrarily assign each link with a direction for book-keeping purposes.

Example: For the lattice in Figure 2.1; $q = 2$, $X(0,1) = [0,0]^{\mathrm{t}}$, $X(0,2) = [1/2, 1/2]^{\mathrm{t}}$, and $\mathbb{B}_+ = \{\, (1,\ [1,0],\ 1),\ (1,\ [0,1],\ 1),\ (1,\ [0,0],\ 2),\ (2,\ [1,0],\ 1),\ (2,\ [1,1],\ 1)\,\}$. $\square$

2.3. **Functions on lattices.** With each node $(m,\kappa)\in\Omega$ we associate a "potential" $\boldsymbol{u}(m,\kappa)\in\mathbb{C}^r$ that represents a quantity such as a temperature or a displacement. We let $\mathcal{V}$ denote the set of all functions $\boldsymbol{u}:\ \Omega\to\mathbb{C}^r$, while $l^2(\Omega,\mathbb{C}^r)=l^2(\Omega)$ denotes the subset of square summable functions. This is a Hilbert space with the inner product

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle := \sum_{(m,\kappa)\in\Omega}\overline{\boldsymbol{u}(m,\kappa)}\cdot\boldsymbol{v}(m,\kappa).$$

We let $\mathcal{V}_J := l^2(\Omega_J,\mathbb{C}^r)$ denote the functions in $\mathcal{V}$ that are supported in Ω_J, and define P_J as the canonical projection $\mathcal{V}\to\mathcal{V}_J$. (Throughout the paper, we use the bold font $\boldsymbol{u},\boldsymbol{v},\boldsymbol{f},\dots$ to represent functions that are defined on integer lattices.)

The potential associated with a cell can either be viewed as a collection of q vectors with r elements, or as a single vector with qr elements,

$$\begin{aligned}(2.1)\quad \boldsymbol{u}(m) &= [\boldsymbol{u}(m,1)^{\rm t},\dots,\boldsymbol{u}(m,q)^{\rm t}]^{\rm t}\\ &= \big[\boldsymbol{u}_1(m,1),\dots,\boldsymbol{u}_r(m,1),\boldsymbol{u}_1(m,2),\dots,\boldsymbol{u}_r(m,q{-}1),\boldsymbol{u}_1(m,q),\dots,\boldsymbol{u}_r(m,q)\big]^{\rm t}\in\mathbb{C}^{qr}.\end{aligned}$$

We shall frequently make use of this trivial identification of $l^2(\mathbb{Z}^d\times\mathbb{N}_q,\mathbb{C}^r)$ and $l^2(\mathbb{Z}^d,\mathbb{C}^{qr})$.

2.4. **The Fourier transform.** We define a discrete Fourier transform via

$$(2.2)\qquad \boldsymbol{F}:\ l^2(\mathbb{Z}^d,\mathbb{C}^{qr})\to L^2(I^d,\mathbb{C}^{qr}):\ \boldsymbol{u}\mapsto\tilde{\boldsymbol{u}}(\xi)=\sum_{m\in\mathbb{Z}^d}e^{\mathbf{i}m\cdot\xi}\boldsymbol{u}(m),$$

where $I^d := (-\pi,\pi)^d$. The equivalent of (2.1) in the Fourier domain is then

$$\begin{aligned}(2.3)\quad \tilde{\boldsymbol{u}}(\xi) = [\tilde{\boldsymbol{u}}(\xi,1)^{\rm t},\dots,\tilde{\boldsymbol{u}}(\xi,q)^{\rm t}]^{\rm t} &=\\ \big[\tilde{\boldsymbol{u}}_1(\xi,1),\dots,\tilde{\boldsymbol{u}}_r(\xi,1),\dots,\tilde{\boldsymbol{u}}_1(\xi,q),\dots,\tilde{\boldsymbol{u}}_r(\xi,q)\big]^{\rm t}&\in\mathbb{C}^{qr}.\end{aligned}$$

The following identity (Parseval) says that $\boldsymbol{F}$ is an isometric isomorphism,

$$(2.4)\qquad ||\boldsymbol{u}||^2_{l^2(\mathbb{Z}^d)}=\sum_{m\in\mathbb{Z}^d}|\boldsymbol{u}(m)|^2=\frac{1}{(2\pi)^d}\int_{I^d}|\tilde{\boldsymbol{u}}(\xi)|^2\,d\xi=\frac{1}{(2\pi)^d}||\tilde{\boldsymbol{u}}||^2_{l^2(I^d)}.$$

3. Basic properties of the lattice equilibrium equation

We refer to an equation that relates a potential field $\boldsymbol{u}$ on $\mathbb{Z}^d$ (representing, *e.g.*, temperatures or displacements) to a load field $\boldsymbol{f}$ (representing, *e.g.*, heat sources or forces) as a lattice equilibrium equation. In Section 3.1 we derive such a global equation from a local equilibrium condition in a framework that is sufficiently general that it covers all the three models under consideration. In Sections 3.2 and 3.3, we prove that the equilibrium operator is bounded, self-adjoint and positive semi-definite as an operator on $l^2(\mathbb{Z}^d)$.

Considered as an operator on $l^2(\mathbb{Z}^d)$, the equilibrium operator is not positive definite in any of the three models that we consider. However, by redefining the function spaces (loosely speaking, we "factor out" the constants and the rigid body motions), it is possible to prove lower bounds on the energy functional that ensure the well-posedness of the equilibrium equations. The details of this analysis, which will be different for each of the three models, are presented in sections 4, 6 and 7.

3.1. **Derivation of the equilibrium equation.** We start by specifying the equilibrium condition for a single link. Consider the link (κ, n, λ) that connects the node $(0, \kappa)$ to the node (n, λ). If the two ends are given potentials $u, v \in \mathbb{C}^r$, then the loads $f, g \in \mathbb{C}^r$ that need to be applied to the two ends to keep the link in equilibrium are specified by a non-negative Hermitian matrix $A^{(\kappa,n,\lambda)} \in \mathbb{C}^{2r\times 2r}$,

$$A^{(\kappa,n,\lambda)} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \quad \text{where} \quad A^{(\kappa,n,\lambda)} = \begin{bmatrix} B^{(\kappa,n,\lambda)} & C^{(\kappa,n,\lambda)} \\ (C^{(\kappa,n,\lambda)})^* & D^{(\kappa,n,\lambda)} \end{bmatrix}. \tag{3.1}$$

Here, $B^{(\kappa,n,\lambda)}$, $C^{(\kappa,n,\lambda)}$ and $D^{(\kappa,n,\lambda)}$ are all $r \times r$ matrices. From $A^{(\kappa,n,\lambda)}$, we define an operator $\boldsymbol{A}^{(\kappa,n,\lambda)}$ acting on the global potential $\boldsymbol{u}$ by

$$\begin{aligned} [\boldsymbol{A}^{(\kappa,n,\lambda)}\boldsymbol{u}](0,\kappa) =& B^{(\kappa,n,\lambda)}\boldsymbol{u}(0,\kappa) + C^{(\kappa,n,\lambda)}\boldsymbol{u}(n,\lambda) \\ [\boldsymbol{A}^{(\kappa,n,\lambda)}\boldsymbol{u}](n,\lambda) =& (C^{(\kappa,n,\lambda)})^*\boldsymbol{u}(0,\kappa) + D^{(\kappa,n,\lambda)}\boldsymbol{u}(n,\lambda), \end{aligned} \tag{3.2}$$

and $[\boldsymbol{A}^{(\kappa,n,\lambda)}\boldsymbol{u}](l,\mu) = 0$, for all other nodes (l,μ). The operator that represents the connection between the nodes (m,κ) and $(m+n,\lambda)$ is then given by $s_{-m}\boldsymbol{A}^{(\kappa,n,\lambda)}s_m$, where for $m \in \mathbb{Z}^d$ the translation operator $s_m : \mathcal{V} \to \mathcal{V}$ is defined by

$$[s_m\boldsymbol{u}](l) := \boldsymbol{u}(l-m).$$

The operator $\boldsymbol{A} : \mathcal{V} \to \mathcal{V}$ that accounts for all links is obtained by adding the contributions from each individual link

$$\boldsymbol{A} := \sum_{m\in\mathbb{Z}^d} \sum_{(\kappa,n,\lambda)\in\mathbb{B}_+} s_{-m}\boldsymbol{A}^{(\kappa,n,\lambda)}s_m, \tag{3.3}$$

whence we obtain the global equilibrium equation

$$[\boldsymbol{A}\boldsymbol{u}](m,\kappa) = \boldsymbol{f}(m,\kappa), \qquad \forall\,(m,\kappa) \in \Omega. \tag{3.4}$$

In the Fourier domain, $\boldsymbol{A}$ is a multiplicative operator;

$$\big[\boldsymbol{F}[\boldsymbol{A}\boldsymbol{u}]\big](\xi) = \left[\sum_{(\kappa,n,\lambda)\in\mathbb{B}_+} \Upsilon^{(\kappa,n,\lambda)}(\xi)\right]\tilde{\boldsymbol{u}}(\xi),$$

where each matrix $\Upsilon^{(\kappa,n,\lambda)}(\xi)$ consists of $q \times q$ blocks, each of size $r \times r$. The four non-zero blocks are located at the intersections of the κ, λ-rows and columns,

$$\Upsilon^{(\kappa,n,\lambda)}(\xi) := \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & B^{(\kappa,n,\lambda)} & \cdots & e^{\mathbf{i}n\cdot\xi}C^{(\kappa,n,\lambda)} & \cdots \\ & \vdots & & \vdots & \\ \cdots & e^{-\mathbf{i}n\cdot\xi}(C^{(\kappa,n,\lambda)})^{\mathrm{t}} & \cdots & D^{(\kappa,n,\lambda)} & \cdots \\ & \vdots & & \vdots & \end{bmatrix}. \tag{3.5}$$

Introducing the symbol $\sigma(\xi) \in \mathbb{C}^{qr\times qr}$ as the Fourier representation of $\boldsymbol{A}$,

$$\sigma(\xi) := \sum_{(\kappa,n,\lambda)\in\mathbb{B}_+} \Upsilon^{(\kappa,n,\lambda)}(\xi) = \boldsymbol{F}\,\boldsymbol{A}\,\boldsymbol{F}^{-1}, \tag{3.6}$$

the equilibrium equation (3.4) can be written equivalently as

$$\sigma(\xi)\tilde{\boldsymbol{u}}(\xi) = \tilde{\boldsymbol{f}}(\xi), \qquad \forall\ \xi \in I^d. \tag{3.7}$$

Examples of how this notation can be used to describe specific lattices and lattice models are given in Appendix A.

3.2. **Basic properties of the equilibrium operator.** Since the index set $\mathbb{B}_+$ in (3.3) is finite, the global operator $\boldsymbol{A}$ is bounded on $l^2(\mathbb{Z}^d \times \mathbb{N}_q)$ and inherits the properties of self-adjointness and positive semi-definiteness from the local matrices $A^{(\kappa,n,\lambda)}$. We have:

Lemma 3.1. *The operator $\boldsymbol{A}$ is bounded, self-adjoint and positive semi-definite on $l^2(\mathbb{Z}^d \times \mathbb{N}_q)$.*

Since $\boldsymbol{F}$ is an isometric isomorphism from $l^2(\mathbb{Z}^d)$ to $L^2(I^d)$, the following result is a direct consequence of Lemma 3.1:

Lemma 3.2. *The symbol $\sigma(\xi)$ is a uniformly bounded Hermitian positive semi-definite matrix.*

3.3. **The quadratic form of $\boldsymbol{A}$.** Given a potential field $\boldsymbol{u}$, we say that the quantity

$$\left\langle s_{-m}\boldsymbol{u}, \boldsymbol{A}^{(\kappa,n,\lambda)} s_{-m}\boldsymbol{u} \right\rangle = \left[\begin{array}{c} \overline{\boldsymbol{u}(m,\kappa)} \\ \overline{\boldsymbol{u}(m+n,\lambda)} \end{array}\right]^{\mathrm{t}} A^{(\kappa,n,\lambda)} \left[\begin{array}{c} \boldsymbol{u}(m,\kappa) \\ \boldsymbol{u}(m+n,\lambda) \end{array}\right]$$

is the "energy" stored in the link connecting the node (m,κ) to the node $(m+n,\lambda)$. We then define

$$W[\boldsymbol{u}](m) := \sum_{(\kappa,n,\lambda)\in\mathbb{B}_+} \left\langle s_{-m}\boldsymbol{u}, \boldsymbol{A}^{(\kappa,n,\lambda)} s_{-m}\boldsymbol{u} \right\rangle \tag{3.8}$$

as the energy of the bars originating in the cell m and let

$$||\boldsymbol{u}||^2_{\boldsymbol{A}} := \lim_{J\to\infty} \sum_{|m|\le J} W[\boldsymbol{u}](m) = \sum_{m\in\mathbb{Z}^d} \sum_{(\kappa,n,\lambda)\in\mathbb{B}_+} \left[\begin{array}{c} \overline{\boldsymbol{u}(m,\kappa)} \\ \overline{\boldsymbol{u}(m+n,\lambda)} \end{array}\right] A^{(\kappa,n,\lambda)} \left[\begin{array}{c} \boldsymbol{u}(m,\kappa) \\ \boldsymbol{u}(m+n,\lambda) \end{array}\right]$$

define the global energy semi-norm. Invoking Parseval's relation (2.4), we find that

$$||\boldsymbol{u}||^2_{\boldsymbol{A}} = \frac{1}{(2\pi)^d} \int_{I^d} \overline{\tilde{\boldsymbol{u}}(\xi)} \cdot \sigma(\xi)\tilde{\boldsymbol{u}}(\xi)\, d\xi.$$

Remark 3.1. When $||\boldsymbol{u}||_{\boldsymbol{A}} < \infty$ it is the case that $||\boldsymbol{u}||^2_{\boldsymbol{A}} = \langle \boldsymbol{u}, \boldsymbol{A}\boldsymbol{u}\rangle =: \langle \boldsymbol{u}, \boldsymbol{A}, \boldsymbol{u}\rangle$, but there typically exist potentials $\boldsymbol{u} \in \mathcal{V}$ such that $\boldsymbol{A}\boldsymbol{u} = 0$ even though $||\boldsymbol{u}||_{\boldsymbol{A}} = \infty$.

We will next show that the quadratic form on $\mathbb{C}^{qr}$ induced by $\sigma(\xi)$, namely

$$\varphi \mapsto \langle \varphi, \sigma(\xi), \varphi\rangle := \overline{\varphi} \cdot [\sigma(\xi)\varphi],$$

is related to the energy of a certain periodic potential field generated by φ and ξ; splitting the vector φ into subvectors, $\varphi = [\varphi(1), \ldots, \varphi(q)]$, where $\varphi(\kappa) \in \mathbb{C}^r$, we define a periodic potential field by setting $\boldsymbol{u}(m,\kappa) := \varphi(\kappa)e^{-\mathbf{i}m\cdot\xi}$. For this displacement field, $W[\boldsymbol{u}](m)$ does not depend on m:

Lemma 3.3. *Given $\xi \in I^d$ and $\varphi \in \mathbb{C}^{qr}$, define a quasi-periodic lattice function by setting $\boldsymbol{u}(m,\kappa) := \varphi(\kappa)e^{-\mathbf{i}m\cdot\xi}$. Then $W[\boldsymbol{u}](m) = \langle \varphi, \sigma(\xi), \varphi\rangle$.*

Proof: Inserting the expression for $\boldsymbol{u}$ in the definition of W, we get

$$W[\boldsymbol{u}](m) = \sum_{(\kappa,n,\lambda)\in\mathbb{B}_+} e^{\mathbf{i}m\cdot\xi} \begin{bmatrix} \overline{\varphi(\kappa)} \\ e^{\mathbf{i}n\cdot\xi}\overline{\varphi(\lambda)} \end{bmatrix} \cdot A^{(\kappa,n,\lambda)} e^{-\mathbf{i}m\cdot\xi} \begin{bmatrix} \varphi(\kappa) \\ e^{-\mathbf{i}n\cdot\xi}\varphi(\lambda) \end{bmatrix}$$
$$= \sum_{(\kappa,n,\lambda)\in\mathbb{B}_+} \overline{\varphi} \cdot [\Upsilon^{(\kappa,n,\lambda)}(\xi)\varphi],$$

which equals $\overline{\varphi} \cdot [\sigma(\xi)\varphi]$ by the definitions (3.5) and (3.6). □

4. The conduction problem

In this section we consider the problem of determining the temperature distribution of a lattice in thermo-static equilibrium. The "potential" of a node is then its temperature, and the "load" is a heat source (both are scalars). In Section 4.1 we describe the mathematical model, in 4.2 we prove Korn's inequality for the infinite lattice, in 4.3 we give conditions on the load and the lattice geometry that are necessary and sufficient for the basic lattice equilibrium equation to be well-posed, and in 4.4 we construct and describe an explicit representation of the inverse operator.

4.1. **The model.** An individual link of conductivity α is said to be in equilibrium if the temperatures of its endpoints, u and v, are related to the fluxes through the endpoints f and g, by $f = -g = \alpha(u - v)$. Thus, for each link $(\kappa, n, \lambda) \in \mathbb{B}_+$, we have, *cf.* (3.1),

$$A^{(\kappa,n,\lambda)} = \alpha^{(\kappa,n,\lambda)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

for some $\alpha^{(\kappa,n,\lambda)} > 0$. The energy functional is simply

$$||\boldsymbol{u}||^2_{\boldsymbol{A}} = \langle \boldsymbol{u}, \boldsymbol{A}, \boldsymbol{u}\rangle = \sum_{m\in\mathbb{Z}^d} \sum_{(\kappa,n,\lambda)\in\mathbb{B}_+} \alpha^{(\kappa,n,\lambda)} |\boldsymbol{u}(m+n,\lambda) - \boldsymbol{u}(m,\kappa)|^2. \tag{4.1}$$

For an explicit example, see Section A.1.

4.2. **Coercivity of the lattice operator.** In this section we prove that if a lattice is connected, then the equilibrium operator $\boldsymbol{A}$ is coercive in the sense that the symbol $\sigma(\xi)$ satisfies Korn's inequality (1.13). In order to give a precise statement, we first need to introduce a change of variables that separates the average potential in a cell from the intra-cell oscillation. We set $\Psi_{\mathrm{a}} := q^{-1/2}[1,\ldots,1]^{\mathrm{t}} \in \mathbb{C}^q$ and let Ψ_{d} be a $q \times (q-1)$ matrix such that $\Psi = [\Psi_{\mathrm{a}}, \Psi_{\mathrm{d}}]$ forms a unitary matrix. Then we split $\varphi \in \mathbb{C}^q$ into an average φ_{a} and a difference φ_{d} as follows,

$$\begin{bmatrix} \varphi_{\mathrm{a}} \\ \varphi_{\mathrm{d}} \end{bmatrix} = \begin{bmatrix} \Psi_{\mathrm{a}}^{\mathrm{t}}\varphi \\ \Psi_{\mathrm{d}}^{\mathrm{t}}\varphi \end{bmatrix} = \Psi^{\mathrm{t}}\varphi.$$

This section is devoted to proving the following result:

Lemma 4.1. *If a lattice is connected, then there exist $c > 0$ and $C < \infty$ such that*

$$c\left(|\xi|^2|\varphi_{\rm a}|^2 + |\varphi_{\rm d}|^2\right) \le \langle \varphi, \sigma(\xi), \varphi\rangle \le C\left(|\xi|^2|\varphi_{\rm a}|^2 + |\varphi_{\rm d}|^2\right), \qquad \forall\ \varphi \in \mathbb{C}^q. \tag{4.2}$$

Remark 4.1. Using Parseval's relation (2.4), we can restate (4.2) in physical space. To this end, define a discrete Laplace operator by

$$[\boldsymbol{A}_0\boldsymbol{v}](m) := \sum_{j=1}^{d}\left(-\boldsymbol{v}(m-e_j) + 2\boldsymbol{v}(m) - \boldsymbol{v}(m+e_j)\right), \tag{4.3}$$

where the e_j's are the canonical unit vectors of $\mathbb{C}^d$ (so that for instance $e_1 = [1, 0, \ldots, 0]^{\rm t}$). The symbol of $\boldsymbol{A}_0$ is

$$\sigma_0(\xi) = \sum_{j=1}^{d} 4\sin^2(\xi_j/2).$$

Since $\sigma_0(\xi) \sim |\xi|^2$ on I^d, the inequalities (4.2) are equivalent with

$$c\left(\langle \boldsymbol{u}_{\rm a}, \boldsymbol{A}_0, \boldsymbol{u}_{\rm a}\rangle + ||\boldsymbol{u}_{\rm d}||^2\right) \le \langle \boldsymbol{u}, \boldsymbol{A}, \boldsymbol{u}\rangle \le C\left(\langle \boldsymbol{u}_{\rm a}, \boldsymbol{A}_0, \boldsymbol{u}_{\rm a}\rangle + ||\boldsymbol{u}_{\rm d}||^2\right), \tag{4.4}$$

where $\boldsymbol{u}_{\rm a}(m) = \Psi_{\rm a}^{\rm t}\boldsymbol{u}(m)$ and $\boldsymbol{u}_{\rm d} = \Psi_{\rm d}^{\rm t}\boldsymbol{u}(m)$.

We will present a proof of Lemma 4.1 that is slightly longer than necessary but that has the merit of being readily generalizable to more complicated lattice models. First we will show that for scalar problems, the connectivity property is equivalent to a more general property regarding projections of nullspaces onto finite sub-trusses (this non-degeneracy condition turns out to be generalizable to mechanical problems). Then we give two lemmas that show that the new non-degeneracy condition implies certain "proto" upper and lower bounds on $\langle \cdot, \sigma(\xi), \cdot\rangle$. Once this is done, we shall prove Lemma 4.1.

From (4.1), it is clear that a lattice is connected if and only if the null-space of $||\cdot||_{\boldsymbol{A}}$ consists of the set of constant functions, which we call $\mathcal{N}$. To enable us to use a compactness argument later, we want to reformulate this condition as a condition on finite sub-trusses. We let $\boldsymbol{A}_J$ denote the operator corresponding to all links that originate in the box Ω_J,

$$\boldsymbol{A}_J := \sum_{|m|_\infty \le J}\ \sum_{(\kappa,n,\lambda)\in\mathbb{B}_+} s_{-m}\boldsymbol{A}^{(\kappa,n,\lambda)}s_m.$$

For a connected lattice, we can always find a J such that the nodes in the set Ω_1 are connected to one another through links that contribute to $\boldsymbol{A}_J$, as shown in Figure 4.1. Another way of saying that all the nodes in Ω_1 are connected is to say that if $\boldsymbol{u} \in \text{Null}(\boldsymbol{A}_J)$, then $\boldsymbol{u}$ must be constant on Ω_1, and thus $\boldsymbol{u} \in P_{\Omega_1}\mathcal{N}$. We have now established that the following statements are equivalent:

(1) The lattice is connected.
(2) $\text{Null}(||\cdot||_{\boldsymbol{A}}) = \mathcal{N}$.
(3) There exists an integer J such that $P_{\Omega_1}\text{Null}(\boldsymbol{A}_J) = P_{\Omega_1}\mathcal{N}$.

Using the equivalence of (1) and (3), we prove the following lemma:

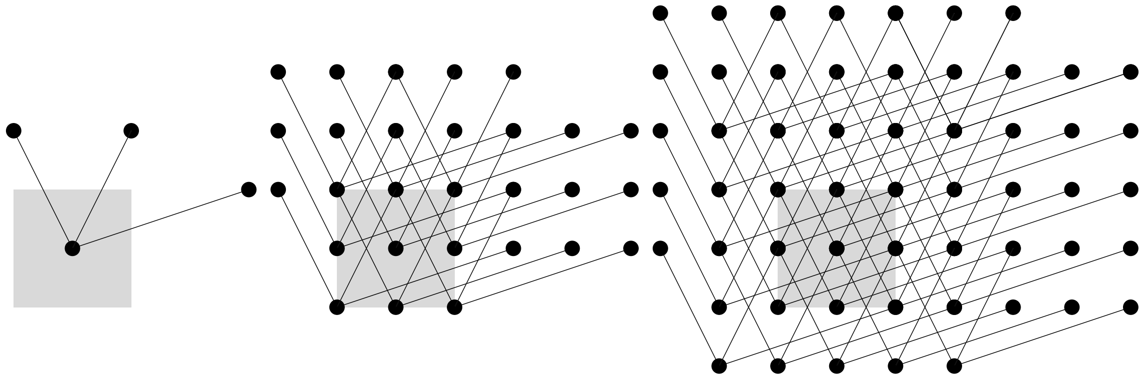

FIGURE 4.1. The three sub-lattices corresponding to $\boldsymbol{A}_0$, $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ for a mono-atomic lattice generated by $\mathbb{B}_+ = \{(1,[-1,2],1),(1,[1,2],1),(1,[3,1],1)\}$. Note that for the $\boldsymbol{A}_1$ sub-lattice, the middle node is not connected to any other nodes in Ω_1 (shaded) but that for the $\boldsymbol{A}_2$ sub-lattice, all nodes in Ω_1 are connected.

Lemma 4.2. *If a lattice is connected, then there exists a positive constant c such that, with $\boldsymbol{u}(m,\kappa) = \varphi(\kappa)e^{-\mathbf{i}m\cdot\xi}$,*

$$\langle \varphi, \sigma(\xi), \varphi \rangle \geq c \inf_{\boldsymbol{v}\in\mathcal{N}} ||\boldsymbol{u}-\boldsymbol{v}||^2_{V_1}.$$

Proof: Lemma 3.3 implies that $\langle \varphi, \sigma(\xi), \varphi \rangle = (2J+1)^{-d} \langle \boldsymbol{u}, \boldsymbol{A}_J, \boldsymbol{u} \rangle$. Next we will show that there exists a $c > 0$ such that $\langle \boldsymbol{u}, \boldsymbol{A}_J, \boldsymbol{u} \rangle \geq c||Q\boldsymbol{u}||^2$, where Q denotes the projection onto the orthogonal complement of $\mathcal{N}$ in V_1 (so that $\inf_{\boldsymbol{v}\in\mathcal{N}} ||\boldsymbol{u}-\boldsymbol{v}||_{V_1} = ||Q\boldsymbol{u}||$). Set

$$c := \inf_{\boldsymbol{u}\in\mathcal{V}} \frac{\langle \boldsymbol{u}, \boldsymbol{A}_J, \boldsymbol{u} \rangle}{||Q\boldsymbol{u}||^2}.$$

Now note that the infimum can be restricted to the compact set $\{\boldsymbol{u} \in V_{J+M} : ||\boldsymbol{u}|| = 1\}$, with the M defined in Lemma 4.3. Since the infimum is taken over a compact set, we know that if $c = 0$, then there must exist a minimizer $\boldsymbol{u}'$ such that $||Q\boldsymbol{u}'|| \neq 0$ and $\langle \boldsymbol{u}', \boldsymbol{A}_J, \boldsymbol{u}' \rangle = 0$, but this contradicts statement (3) above. Since we assumed connectivity, we must then have $c > 0$. □

The proto upper bound is straight-forward:

Lemma 4.3. *Let M be the length of the longest link, $M := \max\{|n|_\infty : (\kappa, n, \lambda) \in \mathbb{B}_+\}$. Then there exists a constant C such that, with $\boldsymbol{u}(m,\kappa) = \varphi(\kappa)e^{-\mathbf{i}m\cdot\xi}$,*

$$\langle \varphi, \sigma(\xi), \varphi \rangle \leq C \inf_{\boldsymbol{v}\in\mathcal{N}} ||\boldsymbol{u}-\boldsymbol{v}||^2_{V_M}.$$

Proof: Fix any $\boldsymbol{v} \in \mathcal{N}$ and invoke Lemma 3.3 to obtain that $\langle \varphi, \sigma(\xi), \varphi \rangle = W[\boldsymbol{u}](0) = W[\boldsymbol{u}-\boldsymbol{v}](0)$. Since all links originating in Ω_0 are wholly contained in Ω_M, we know that $W[\boldsymbol{u}-\boldsymbol{v}](0) \leq ||\boldsymbol{A}||_V ||\boldsymbol{u}-\boldsymbol{v}||^2_{V_M}$, and the claim follows from the boundedness of $\boldsymbol{A}$. □

Combining lemmas 4.2 and 4.3 we are now in a position to prove Lemma 4.1.

Proof of Lemma 4.1: Let $\boldsymbol{u}(m,\kappa) := \varphi(\kappa)e^{-\mathbf{i}m\cdot\xi}$, as in Lemmas 4.2 and 4.3.

To prove the upper bound in (4.2), use Lemma 4.3 with the choice $\boldsymbol{v}(m,\kappa) := \varphi_{\mathrm{a}}$ to get

$$\langle\varphi,\sigma(\xi),\varphi\rangle \le C\left(||\boldsymbol{u}_{\mathrm{a}}-\boldsymbol{v}||^2_{V_M} + ||\boldsymbol{u}_{\mathrm{d}}||^2_{V_M}\right)$$
$$= C\sum_{|m|\le M}\left(|\varphi_{\mathrm{a}}e^{-\mathbf{i}m\cdot\xi}-\varphi_{\mathrm{a}}|^2 + |\varphi_{\mathrm{d}}e^{-\mathbf{i}m\cdot\xi}|^2\right) \le C\left(|\xi|^2|\varphi_{\mathrm{a}}|^2 + |\varphi_{\mathrm{d}}|^2\right).$$

Next we prove the lower bound in (4.2) using Lemma 4.2. Set

$$l_{\mathrm{a}}(\xi) := \inf_{\varphi\in\mathbb{C}^q}\frac{1}{2}\frac{\langle\varphi,\sigma(\xi),\varphi\rangle}{|\varphi_{\mathrm{a}}|^2}, \quad \text{and} \quad l_{\mathrm{d}}(\xi) := \inf_{\varphi\in\mathbb{C}^q}\frac{1}{2}\frac{\langle\varphi,\sigma(\xi),\varphi\rangle}{|\varphi_{\mathrm{d}}|^2}, \tag{4.5}$$

so that

$$\langle\varphi,\sigma(\xi),\varphi\rangle \ge l_{\mathrm{a}}(\xi)|\varphi_{\mathrm{a}}|^2 + l_{\mathrm{d}}(\xi)|\varphi_{\mathrm{d}}|^2.$$

First note that the infimum can be restricted to the unit ball in $\mathbb{C}^q$, which is a compact set. Then, since $\boldsymbol{u}\notin\mathcal{N}$ for $\xi\neq 0$, Lemma 4.2 immediately tells us that both $l_{\mathrm{a}}(\xi)$ and $l_{\mathrm{d}}(\xi)$ are positive for $\xi\neq 0$. It remains to investigate their behavior near the origin.

To prove that $l_{\mathrm{d}}(0) > 0$, we note that for $\xi = 0$, the minimizer in Lemma 4.2 is $\boldsymbol{u}_{\mathrm{a}}$, so that

$$\langle\varphi,\sigma(0),\varphi\rangle \ge c||\boldsymbol{u}-\boldsymbol{u}_{\mathrm{a}}||^2_{V_1} = c||\boldsymbol{u}_{\mathrm{d}}||^2_{V_1} = c3^d|\varphi_{\mathrm{d}}|^2.$$

In order to prove that $l_{\mathrm{a}}(\xi)\ge c|\xi|^2$, we first use that

$$\langle\varphi,\sigma(\xi),\varphi\rangle \ge c\inf_{\boldsymbol{v}\in\mathcal{N}}||\boldsymbol{u}-\boldsymbol{v}||^2_{V_1} \ge c\inf_{\boldsymbol{v}\in\mathcal{N}}||\boldsymbol{u}_{\mathrm{a}}-\boldsymbol{v}_{\mathrm{a}}||^2_{V_1} = c\inf_{z\in\mathbb{C}}\sum_{|m|\le 1}|\varphi_{\mathrm{a}}e^{-\mathbf{i}m\cdot\xi}-z|^2.$$

The minimizer is

$$z^{(\min)} = \frac{1}{3^d}\sum_{|m|\le 1}\varphi_{\mathrm{a}}e^{-\mathbf{i}m\cdot\xi} = \varphi_{\mathrm{a}}\prod_{j=1}^{d}\sum_{m_j=-1}^{1}\frac{e^{-\mathbf{i}m_j\xi_j}}{3} = \varphi_{\mathrm{a}}\prod_{j=1}^{d}\left(1-\frac{4}{3}\sin^2\frac{\xi_j}{2}\right) =: \varphi_{\mathrm{a}}\big(1+r_d(\xi)\big),$$

where $r_d(\xi)$ satisfies $|r_d(\xi)|\le C|\xi|^2$. Thus, using that $|a+b|^2\ge |a|^2-|b|^2$ we find that

$$\langle\varphi,\sigma(\xi),\varphi\rangle \ge c\sum_{|m|\le 1}|\varphi_{\mathrm{a}}e^{-\mathbf{i}m\cdot\xi}-\varphi_{\mathrm{a}}-\varphi_{\mathrm{a}}r_d(\xi)|^2$$
$$\ge c|\varphi_{\mathrm{a}}|^2\sum_{|m|\le 1}\left(|e^{-\mathbf{i}m\cdot\xi}-1|^2-|r_d(\xi)|^2\right) \ge c|\varphi_{\mathrm{a}}|^2\left(|\xi|^2-|\xi|^4\right) \ge c|\varphi_{\mathrm{a}}|^2|\xi|^2,$$

which proves that $l_{\mathrm{a}}(\xi)\ge c|\xi|^2$. □

Introducing $L(\xi)$ as a diagonal matrix with the diagonal $[|\xi|^2, 1, \ldots, 1]$ we can reformulate the statement of Lemma 4.1 as $\sigma(\xi)\sim\Psi L(\xi)\Psi^{\mathrm{t}}$. Invoking Lemma 2.1 we then immediately obtain:

Corollary 4.4. *For the conduction problem on a connected lattice,* $\det\sigma(\xi)\sim|\xi|^2$.

4.3. **Well-posedness of the lattice equation.** In this section we will use the coercivity result of Lemma 4.1 to prove that the lattice equilibrium equation

$$\left\{ \begin{aligned} \boldsymbol{A}\boldsymbol{u} &= \boldsymbol{f}, \\ ||\boldsymbol{u}||_{\boldsymbol{A}} &< \infty, \end{aligned} \right. \tag{4.6}$$

is well-posed under certain conditions on $\boldsymbol{f}$. In order to state these conditions, it will be necessary to split $\boldsymbol{f}$ into an average component, $\boldsymbol{f}_{\mathrm{a}}(m) = (1/\sqrt{q}) \sum_{\kappa=1}^{q} \boldsymbol{f}(m,\kappa)$, and an oscillatory component, $\boldsymbol{f}_{\mathrm{d}}(m,\kappa) = \boldsymbol{f}(m,\kappa) - \boldsymbol{f}_{\mathrm{a}}(m)$, since it turns out that stronger decay conditions are required for the $\boldsymbol{f}_{\mathrm{a}}$ than for $\boldsymbol{f}_{\mathrm{d}}$. Moreover, it will be necessary to distinguish between the case of two-dimensional lattices on the one hand, and lattices in higher dimensions on the other, since in the former case, the load $\boldsymbol{f}$ must sum to zero. The principal result of this section is the following:

Theorem 4.5. *(a) In two dimensions, suppose that $|m|\boldsymbol{f}_{\mathrm{a}}(m) \in l^1$, $\sum \boldsymbol{f}(m,\kappa) = 0$, $\boldsymbol{f}_{\mathrm{d}} \in l^2$, and that the lattice is connected. Then (4.6) has a solution $\boldsymbol{u} \in \mathcal{V}$ which is unique up to a constant. This solution satisfies $||\boldsymbol{u}||_{\boldsymbol{A}} \le C\,(||m\boldsymbol{f}_{\mathrm{a}}(m)||_{l^1} + ||\boldsymbol{f}_{\mathrm{d}}||_{l^2})$.*
(b) In three (and higher) dimensions, suppose that $\boldsymbol{f}_{\mathrm{a}} \in l^1$, $\boldsymbol{f}_{\mathrm{d}} \in l^2$, and that the lattice is connected. Then (4.6) has a solution $\boldsymbol{u} \in \mathcal{V}$ which is unique up to a constant. This solution satisfies $||\boldsymbol{u}||_{\boldsymbol{A}} \le C\,(||\boldsymbol{f}_{\mathrm{a}}||_{l^1} + ||\boldsymbol{f}_{\mathrm{d}}||_{l^2})$.

In order to prove Theorem 4.5, we will first prove a more general result whose conditions are stated in terms of the following function spaces (which are lattice equivalents of continuum Sobolev spaces):

$$S_k^r := \{\boldsymbol{v} \in \mathcal{V} :\ ||\boldsymbol{v}||_{S_k^r} < \infty\}, \quad \text{where} \quad ||\boldsymbol{v}||_{S_k^r} = \left[\int_{I^d} \Big(|\xi|^{-k}|\tilde{\boldsymbol{v}}(\xi)|\Big)^r \, d\xi\right]^{1/r}.$$

The general well-posedness result (from which Theorem 4.5 follows) is then:

Theorem 4.6. *Suppose that $\boldsymbol{f}_{\mathrm{a}} \in S_1^2$, $\boldsymbol{f}_{\mathrm{d}} \in S_0^2$, and that the lattice is connected. Then equation (4.6) has a solution $\mathbf{u}$ which is unique up to a constant. This solution satisfies $||\boldsymbol{u}||_{\boldsymbol{A}} \le C\big(||\boldsymbol{f}_{\mathrm{a}}||_{S_1^2} + ||\boldsymbol{f}_{\mathrm{d}}||_{S_0^2}\big)$.*

Proof: The idea of the proof is to construct an energy space (similar to H^1 in the analysis of Poisson's equation), with $\langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}\rangle$ as the inner product. Then we reformulate (4.6) in variational form and use Riesz' representation theorem to prove existence and uniqueness of the solutions.

The first step is to construct the energy space. To this end, let $\boldsymbol{u}$ and $\boldsymbol{v}$ be compactly supported functions in $\mathcal{V}$ and define

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{A}} := \langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}\rangle = \frac{1}{(2\pi)^d} \int_{I^d} \overline{\tilde{\boldsymbol{u}}(\xi)} \cdot \sigma(\xi) \tilde{\boldsymbol{v}}(\xi)\, d\xi,$$

and the corresponding semi-norm $||\boldsymbol{u}||_{\boldsymbol{A}} = \sqrt{\langle \boldsymbol{u}, \boldsymbol{u}\rangle_{\boldsymbol{A}}}$. By Lemma 4.1, we know that for some $c > 0$,

$$||\boldsymbol{u}||_{\boldsymbol{A}}^2 \ge c \int_{I^d} \big(|\xi|^2 |\tilde{\boldsymbol{u}}_{\mathrm{d}}(\xi)|^2 + |\tilde{\boldsymbol{u}}_{\mathrm{a}}(\xi)|^2\big)\, d\xi. \tag{4.7}$$

Then define the energy space W as the closure of the compactly supported functions in $\mathcal{V}$ under the norm $||\cdot||_{\boldsymbol{A}}$. We note that any two functions in $\mathcal{V}$ whose difference is constant are considered identical in W.

Next, we reformulate (4.6) as a variational equation

$$\text{Find } \boldsymbol{u} \in W \text{ such that } \langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{A}} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \text{ for every } \boldsymbol{v} \in W. \tag{4.8}$$

Since there are no non-empty sets of measure zero in Ω, (4.6) and (4.8) are exactly equivalent.

By virtue of Riesz' representation theorem, (4.8) has a unique solution in W provided that the map $\boldsymbol{v} \mapsto \langle \boldsymbol{f}, \boldsymbol{v}\rangle$ is a bounded functional on W. We have

$$\begin{aligned} |\langle \boldsymbol{f}, \boldsymbol{v}\rangle| &= \left| \frac{1}{(2\pi)^d} \int_{I^d} \overline{\tilde{\boldsymbol{f}}(\xi)} \cdot \tilde{\boldsymbol{v}}(\xi)\, d\xi \right| \leq C \int_{I^d} \left(|\tilde{\boldsymbol{f}}_{\mathrm{a}}(\xi)|\, |\tilde{\boldsymbol{v}}_{\mathrm{a}}(\xi)| + |\tilde{\boldsymbol{f}}_{\mathrm{d}}(\xi)|\, |\tilde{\boldsymbol{v}}_{\mathrm{d}}(\xi)| \right) d\xi \\ &\leq C \left[\int_{I^d} \frac{|\tilde{\boldsymbol{f}}_{\mathrm{a}}(\xi)|^2}{|\xi|^2}\, d\xi \right]^{1/2} \left[\int_{I^d} |\xi|^2 |\tilde{\boldsymbol{v}}_{\mathrm{a}}(\xi)|^2\, d\xi \right]^{1/2} + ||\boldsymbol{f}_{\mathrm{d}}||_{l^2}\, ||\boldsymbol{v}_{\mathrm{d}}||_{l^2} \\ &\leq C \left(||\boldsymbol{f}_{\mathrm{a}}||^2_{S^2_1} + ||\boldsymbol{f}_{\mathrm{d}}||^2_{S^2_0} \right)^{1/2} ||\boldsymbol{v}||_W, \end{aligned}$$

where the crucial inequality (4.7) was invoked at the last step.

It only remains to prove that uniqueness in W implies uniqueness up to constants in $\mathcal{V}$. To see this, suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathcal{V}$ both solve (4.6). Then $\boldsymbol{u} - \boldsymbol{v} \in W$ and $||\boldsymbol{u} - \boldsymbol{v}||_{\boldsymbol{A}} = 0$. Since the lattice is connected, equation (4.1) then implies that $\boldsymbol{u} - \boldsymbol{v}$ is constant. □

Theorem 4.5 follows directly from Theorem 4.6 by virtue of the following lemmas:

Lemma 4.7. *If $kr < d$, then $||\cdot||_{S^r_k} \leq C||\cdot||_{l^1}$ and thus $l^1(\Omega) \subseteq S^r_k$.*

Proof: If $\boldsymbol{v} \in l^1$, then $|\tilde{\boldsymbol{v}}(\xi)| \leq C||\boldsymbol{v}||_{l^1}$, and thus

$$||\boldsymbol{v}||^r_{S^r_k} = \int_{I^d} (|\xi|^{-k} |\tilde{\boldsymbol{v}}(\xi)|)^r\, d\xi \leq C \int_0^{\sqrt{d}\pi} (\rho^{-k} ||\boldsymbol{v}||_{l^1})^r \rho^{d-1}\, d\rho = C||\boldsymbol{v}||^r_{l^1} \int_0^{\sqrt{d}\pi} \rho^{d-kr-1}\, d\rho,$$

which is finite if $d - kr > 0$. □

Lemma 4.8. *A sufficient condition for $\boldsymbol{v} \in S^r_k$ is that for some integer $l > k - d/r$ the following conditions hold:*

- **A:** *$\sum (1 + |m|^l) |\boldsymbol{v}(m)| < \infty$.*
- **B:** *$\sum m^\alpha \boldsymbol{v}(m) = 0$, for every $\alpha \in \mathbb{N}^d$ such that $\alpha_1 + \cdots + \alpha_d < l$.*

Proof: Condition **B** implies that $\tilde{\boldsymbol{v}}$ has l vanishing moments. Then, since condition **A** implies that $\tilde{\boldsymbol{v}}$ has l continuous derivatives we find that $|\tilde{\boldsymbol{v}}(\xi)| \leq C|\xi|^l$, whence

$$||\boldsymbol{v}||^r_{S^r_k} = \int_{I^d} (|\xi|^{-k} |\tilde{\boldsymbol{v}}(\xi)|)^r\, d\xi \leq C \int_{I^d} (|\xi|^{-k} |\xi|^l)^r\, d\xi \leq C \int_0^{\sqrt{d}\pi} \rho^{r(l-k)} \rho^{d-1}\, d\rho,$$

which is finite if $r(l - k) + d > 0$. □

4.4. **Properties of $\sigma(\xi)^{-1}$.** In the Fourier domain, the solution of the lattice equilibrium equation has the simple form

$$\tilde{\boldsymbol{u}}(\xi) = \sigma(\xi)^{-1}\tilde{\boldsymbol{f}}(\xi). \tag{4.9}$$

Since $\sigma(\xi)^{-1}$ is analytic and bounded away from the origin, the long-range behavior of $\boldsymbol{u}$ is entirely determined by the singularity at the origin. The following theorem, which characterizes this singularity, is the main result of this section:

Theorem 4.9. *For the conduction problem on a connected lattice, all the entries of $\sigma(\xi)^{-1}$ have the same asymptotic behavior at the origin. To be precise, as $|\xi| \to 0$,*

$$\sigma(\xi)^{-1} = \Psi_{\mathrm{a}}\sigma^{(0)}(\xi)^{-1}\Psi_{\mathrm{a}}^{\mathrm{t}} + O(|\xi|^{-1}),$$

where $\sigma^{(0)}(\xi) = \xi \cdot M\xi$ for some positive definite matrix M.

Proof: The idea of the proof is to combine Lemma 2.1 with the inequality (4.2), which we rewrite as (recall that the notation $A \sim B$ was defined in Section 2.1)

$$\Psi^{\mathrm{t}}\sigma(\xi)\Psi \sim \begin{bmatrix} |\xi|^2 & 0 \\ 0 & I_{q-1} \end{bmatrix}, \tag{4.10}$$

where I_{q-1} is the $(q-1)$-dimensional identity matrix. Invoking Lemma 2.1 we then find that

$$\Psi^{\mathrm{t}}\sigma(\xi)^{-1}\Psi \sim \begin{bmatrix} |\xi|^{-2} & 0 \\ 0 & I_{q-1} \end{bmatrix}. \tag{4.11}$$

To obtain the precise statements of Theorem 4.9 from this inequality, we split the matrices on the left hand sides of (4.10) and (4.11) into cell-wise averages and differences; for $\alpha, \beta \in \{\mathrm{a}, \mathrm{d}\}$, set $\sigma_{\alpha\beta}(\xi) := \Psi_\alpha^{\mathrm{t}}\sigma(\xi)\Psi_\beta$ and $\tau_{\alpha\beta}(\xi) := \Psi_\alpha^{\mathrm{t}}\sigma(\xi)^{-1}\Psi_\beta$ so that

$$\Psi^{\mathrm{t}}\sigma(\xi)\Psi = \begin{bmatrix} \sigma_{\mathrm{aa}}(\xi) & \sigma_{\mathrm{ad}}(\xi) \\ \sigma_{\mathrm{da}}(\xi) & \sigma_{\mathrm{dd}}(\xi) \end{bmatrix}, \quad \text{and} \quad \Psi^{\mathrm{t}}\sigma(\xi)^{-1}\Psi = \begin{bmatrix} \tau_{\mathrm{aa}}(\xi) & \tau_{\mathrm{ad}}(\xi) \\ \tau_{\mathrm{da}}(\xi) & \tau_{\mathrm{dd}}(\xi) \end{bmatrix}. \tag{4.12}$$

The inequality (4.11) implies that

$$|\tau_{\mathrm{aa}}(\xi)| \le C|\xi|^{-2}, \quad |\tau_{\mathrm{ad}}(\xi)| \le C|\xi|^{-1}, \quad |\tau_{\mathrm{da}}(\xi)| \le C|\xi|^{-1}, \quad |\tau_{\mathrm{dd}}(\xi)| \le C. \tag{4.13}$$

Thus, $\sigma(\xi)^{-1} = \Psi_{\mathrm{a}}\tau_{\mathrm{aa}}(\xi)\Psi_{\mathrm{a}}^{\mathrm{t}} + O(|\xi|^{-1})$.

It remains to prove that $\tau_{\mathrm{aa}}(\xi) = (\xi \cdot M\xi)^{-1} + O(|\xi|^{-1})$. To this end, note that

$$\tau_{\mathrm{aa}}(\xi) = \left[\sigma_{\mathrm{aa}}(\xi) - \sigma_{\mathrm{ad}}(\xi)\big(\sigma_{\mathrm{dd}}(\xi)^{-1}\big)\sigma_{\mathrm{da}}(\xi)\right]^{-1}.$$

Equation (4.10) implies that

$$c|\xi|^2 \le \sigma_{\mathrm{aa}}(\xi) \le C|\xi|^2, \quad ||\sigma_{\mathrm{ad}}(\xi)|| = ||\sigma_{\mathrm{da}}(\xi)|| \le C|\xi|, \quad cI_{q-1} \le \sigma_{\mathrm{dd}}(\xi) \le CI_{q-1}. \tag{4.14}$$

This enables us to define

$$\sigma^{(0)}(\xi) := \left[\lim_{\varepsilon\to 0} \varepsilon^2\tau_{\mathrm{aa}}(\varepsilon\xi)\right]^{-1} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left[\sigma_{\mathrm{aa}}(\varepsilon\xi) - \sigma_{\mathrm{ad}}(\varepsilon\xi)\sigma_{\mathrm{dd}}(0)^{-1}\sigma_{\mathrm{da}}(\varepsilon\xi)\right], \tag{4.15}$$

and be assured that $\sigma^{(0)}(\xi) = \xi \cdot M\xi$ for some symmetric matrix M. The first inequality of (4.13) implies that this matrix must be positive definite. $\square$

We can now proceed in two ways: (i) Use the frame-work of generalized functions to construct a fundamental solution $\boldsymbol{G} = \boldsymbol{F}^{-1}[\sigma^{-1}]$, determine its properties using Theorem 4.9 and then make inferences about $\boldsymbol{u}$ from the formula $\boldsymbol{u} = \boldsymbol{G} * \boldsymbol{f}$, see [21]. (ii) View $\tilde{\boldsymbol{u}}(\xi)$ as an L^2-function and use Theorem 4.9 in combination with Paley-Wiener analysis to determine the long range behavior of $\boldsymbol{u}$. We will not go into details, but the main findings from either line of investigation are these:

- Oscillatory components in a potential decay faster (by a factor of $|m|^{-1}$) than the average potential away from a concentrated load.
- The contribution from an oscillatory load decays faster (by a factor of $|m|^{-1}$) than the contribution from a smooth load of the same magnitude.

Both of these statements agree with our expectation that the equilibrium equation should share qualitative behavior with an elliptic partial differential equation.

Remark 4.2. Theorem 4.9 is also central to the analysis of the asymptotic behavior of solutions to the lattice equilibrium equation. As an example, it will turn out that an approximate solution of the lattice equation can be obtained by solving the continuum equation $-\nabla \cdot M\nabla u = f$. See [19] for details.

Remark 4.3. Another application of Theorem 4.9 is that it allows us to give specific condition on $\boldsymbol{f}$ under which the explicit inversion formula

$$\boldsymbol{u}(m) = \left[\boldsymbol{F}^{-1}\left[\sigma^{-1}\tilde{\boldsymbol{f}}\,\right]\right](m) = \frac{1}{(2\pi)^d}\int_{I^d} e^{-\mathbf{i}m\cdot\xi}\sigma(\xi)^{-1}\tilde{\boldsymbol{f}}(\xi)\,d\xi, \tag{4.16}$$

is absolutely integrable. From the inequalities (4.13) it follows that

$$\begin{aligned}|\sigma(\xi)^{-1}\tilde{\boldsymbol{f}}(\xi)| &\leq (|\tau_{\mathrm{aa}}(\xi)| + |\tau_{\mathrm{da}}(\xi)|)|\tilde{\boldsymbol{f}}_{\mathrm{a}}(\xi)| + (|\tau_{\mathrm{ad}}(\xi)| + |\tau_{\mathrm{dd}}(\xi)|)|\tilde{\boldsymbol{f}}_{\mathrm{d}}(\xi)| \\ &\leq C|\xi|^{-2}|\tilde{\boldsymbol{f}}_{\mathrm{a}}(\xi)| + C|\xi|^{-1}|\tilde{\boldsymbol{f}}_{\mathrm{d}}(\xi)|.\end{aligned}$$

Thus, $e^{-\mathbf{i}m\cdot\xi}\sigma(\xi)^{-1}\tilde{\boldsymbol{f}}(\xi) \in L^1(I^d)$ if $\boldsymbol{f}_{\mathrm{a}} \in S_1^2$ and $\boldsymbol{f}_{\mathrm{d}} \in S_1^1$. For the special case of a mono-atomic cubic lattice, this result was reported by Duffin [8].

5. General coercivity results

The findings for conduction problems in Section 4 will in sections 6 and 7 be generalized to mechanical problems of truss and frame type, respectively. Before giving those results, it simplifies matters to restate some of the results for conduction problems in a more general setting. We let $\mathcal{N} \subset \mathcal{V}$ denote a set of function which we know is in the nullset of $||\cdot||_{\boldsymbol{A}}$ and that we want to prove constitutes the entire nullset. For conduction problems, $\mathcal{N}$ was the set of constants, for mechanical problems, $\mathcal{N}$ will be the rigid body motions. We will consider only nullspaces $\mathcal{N}$ of finite dimension. The periodicity of the lattice then implies that $\mathcal{N}$ must satisfy:

A: Translation invariance: $s_m\mathcal{N} = \mathcal{N}$, for any $m \in \mathbb{Z}^d$.

B: Localization: there exists an integer k such that any $\boldsymbol{v} \in \mathcal{N}$ is uniquely determined by $P_k\boldsymbol{v}$.

We will next define a local coercivity condition that is a generalization of condition (3) given before Lemma 4.2. This condition is sufficient for the Korn-type inequality (1.13) to hold and it has the virtue of being easily checked computationally, once a lattice and an interaction model are given.

Definition: Say that $\boldsymbol{A}$ is $\mathcal{N}$-coercive if there is a finite J such that $P_{\Omega_{k+1}}\mathrm{Null}(\boldsymbol{A}_J) = P_{\Omega_{k+1}}\mathcal{N}$. (We recall that k is the parameter in the condition B above.)

The point of this definition is that the property of $\mathcal{N}$-coercivity is local and can be verified computationally by simply constructing the sequence of matrices $\boldsymbol{A}_J$ and investigating their nullspaces. Our next result states that when $\boldsymbol{A}$ satisfies the (local) $\mathcal{N}$-coercivity property, then $\boldsymbol{A}$ is also globally coercive "modulo $\mathcal{N}$".

Lemma 5.1. *If $\boldsymbol{A}$ is $\mathcal{N}$-coercive, then* $\mathrm{Null}(||\cdot||_{\boldsymbol{A}}) = \mathcal{N}$.

Proof: Suppose that the lattice is $\mathcal{N}$-coercive. By definition, $\mathcal{N} \subseteq \mathrm{Null}(||\cdot||_{\boldsymbol{A}})$, so we only need to prove that $\mathcal{N} \supseteq \mathrm{Null}(||\cdot||_{\boldsymbol{A}})$. Given a $\boldsymbol{v} \in \mathrm{Null}(||\cdot||_{\boldsymbol{A}})$ we find that

$$P_{\Omega_{k+1}}\boldsymbol{v} \in P_{\Omega_{k+1}}\mathrm{Null}(||\cdot||_{\boldsymbol{A}}) \subset P_{\Omega_{k+1}}\mathrm{Null}(||\cdot||_{\boldsymbol{A}_J}) = P_{\Omega_{k+1}}\mathrm{Null}(\boldsymbol{A}_J) = P_{\Omega_{k+1}}\mathcal{N}. \tag{5.1}$$

Statement (5.1) combined with property B imply that there exists a unique $\boldsymbol{v}' \in \mathcal{N}$ such that $P_{\Omega_{k+1}}\boldsymbol{v} = P_{\Omega_{k+1}}\boldsymbol{v}'$. Next we need to show that $\boldsymbol{v}' = \boldsymbol{v}$ on all of Ω. Let $m \in \mathbb{Z}^d$ be any vector such that $|m|_\infty = 1$. By property A, the translated function $s_m\boldsymbol{v}$ also belongs to $\mathrm{Null}(\boldsymbol{A})$. We can therefore repeat the calculation (5.1) to find a unique $\boldsymbol{v}'' \in \mathcal{N}$ that equals $\boldsymbol{v}$ in $s_m\Omega_{k+1}$. Now use that $P_{\Omega_k}\boldsymbol{v}'' = P_{\Omega_k}\boldsymbol{v}'$ and the fact that a function in $\mathcal{N}$ is uniquely determined by its restriction to Ω_k to deduce that $\boldsymbol{v}'' = \boldsymbol{v}'$. Since m was arbitrary, this shows that $\boldsymbol{v}(m) = \boldsymbol{v}'(m)$ for $|m|_\infty \leq k+2$. This process can be continued to cover all of Ω. $\square$

We can now state the proto upper and lower bounds on $\langle\cdot,\sigma(\xi),\cdot\rangle$ in a general framework. They are proved by following the exact same steps taken in the analysis of the conduction case.

Lemma 5.2. *If a lattice is $\mathcal{N}$-coercive, then there exists a positive constant c such that, with $\boldsymbol{u}(m,\kappa) = \varphi(\kappa)e^{-\mathrm{i}m\cdot\xi}$,*

$$\langle\varphi,\sigma(\xi),\varphi\rangle \geq c \inf_{\boldsymbol{v}\in\mathcal{N}} ||\boldsymbol{u}-\boldsymbol{v}||^2_{V_{k+1}}.$$

Lemma 5.3. *Let M be the length of the longest link, $M := \max\{|n|_\infty : (\kappa,n,\lambda) \in \mathbb{B}_+\}$. Then there exists a constant C such that, with $\boldsymbol{u}(m,\kappa) = \varphi(\kappa)e^{-\mathrm{i}m\cdot\xi}$,*

$$\langle\varphi,\sigma(\xi),\varphi\rangle \leq C \inf_{\boldsymbol{v}\in\mathcal{N}} ||\boldsymbol{u}-\boldsymbol{v}||^2_{V_M}.$$

The final result of this section says that $\mathcal{N}$-coercivity is a direct consequence of a lattice being connected for those lattice models that satisfy the following property:

Definition: We say that a link (κ,n,λ) is $\mathcal{N}$-rigid if for any $\varphi \in \mathbb{C}^r$, there exists a unique $\boldsymbol{v} \in \mathcal{N}$ such that $\boldsymbol{v}(0,\kappa) = \varphi$ and $\boldsymbol{A}^{(\kappa,n,\lambda)}\boldsymbol{v} = 0$.

Of the three models that we consider, the conduction model and the mechanical frame model have $\mathcal{N}$-rigid links, but the mechanical truss model does not.

Lemma 5.4. *Suppose that all links are $\mathcal{N}$-rigid. Then if a lattice is connected, it is $\mathcal{N}$-coercive.*

Proof: Let J be large enough that every node in Ω_{k+1} is connected to the node $(0,1)$ through a path entirely contained within Ω_J. Fixing a $\boldsymbol{v} \in \mathrm{Null}(\boldsymbol{A}_J)$ we need to show that there exists a $\boldsymbol{v}' \in \mathcal{N}$ such that $P_{\Omega_{k+1}}\boldsymbol{v} = P_{\Omega_{k+1}}\boldsymbol{v}'$.

Due to the symmetry and the positive semi-definiteness of all the local bars we know that $s_{-m}\boldsymbol{A}^{(\kappa,n,\lambda)}s_m\boldsymbol{v} = 0$ for all $|m| \leq J$. Thus, given any link $(1,n,\lambda)$, there exists a unique $\boldsymbol{v}' \in \mathcal{N}$ such that $\boldsymbol{v}(0,1) = \boldsymbol{v}'(0,1)$ and $\boldsymbol{v}(n,\lambda) = \boldsymbol{v}'(n,\lambda)$.

Given any node $(l,\mu) \in \Omega_{k+1}$ choose a path that connects $(0,1)$ to (l,μ). Since $\boldsymbol{A}_J\boldsymbol{v}' = 0$ and $\boldsymbol{v}'(0,1) = \boldsymbol{v}(0,1)$ it then follows from the $\mathcal{N}$-rigidity of all the links in the path that $\boldsymbol{v}'(l,\mu) = \boldsymbol{v}(l,\mu)$. Thus $P_{\Omega_{k+1}}\boldsymbol{v} = P_{\Omega_{k+1}}\boldsymbol{v}'$, which concludes the proof. □

6. Mechanical trusses

The results regarding the equations of thermo-static equilibrium presented in Section 4 will in this section be extended to the case of elasto-static equilibrium of mechanical lattices that derive their main strength from the axial stiffness of their links, such as lattices B and C in Figure 1.1. For such geometries we ignore the bending stiffnesses of the bars and view the lattice as a collection of axial springs that are pin-jointed at the nodes. The potential of a node is then its translational displacement, and the load is a force. Consequently, the nodal potentials and loads have the same dimension as the surrounding space, $\boldsymbol{u}(m,\kappa), \boldsymbol{f}(m,\kappa) \in \mathbb{R}^d$.

After describing the mathematical model in Section 6.1, we prove Korn's inequality for truss lattices in Section 6.2. This proof is similar to the proof of the corresponding result for the conduction model but some additional considerations are required since the non-degeneracy condition for truss lattices is more involved. Then in sections 6.3 and 6.4 we investigate when the lattice equilibrium equation is well-posed and describe the inverse symbol.

6.1. **The model.**

In order to derive the local stiffness matrix, consider a strut of axial stiffness α that is directed along the unit vector $e \in \mathbb{R}^d$. If the ends of the strut are displaced by the vectors $u, v \in \mathbb{R}^d$, then the forces needed to keep it in equilibrium are $f = \alpha\, e\,(e \cdot (u - v))$ and $g = -f$.Thus, the local stiffness matrix for the link (κ,n,λ) with stiffness $\alpha^{(\kappa,n,\lambda)}$ takes the form

$$A^{(\kappa,n,\lambda)} = \alpha^{(\kappa,n,\lambda)} \left[\begin{array}{rr} e^{(\kappa,n,\lambda)}\big[e^{(\kappa,n,\lambda)}\big]^{\mathrm{t}} & -e^{(\kappa,n,\lambda)}\big[e^{(\kappa,n,\lambda)}\big]^{\mathrm{t}} \\ -e^{(\kappa,n,\lambda)}\big[e^{(\kappa,n,\lambda)}\big]^{\mathrm{t}} & e^{(\kappa,n,\lambda)}\big[e^{(\kappa,n,\lambda)}\big]^{\mathrm{t}} \end{array}\right], \tag{6.1}$$

where $e^{(\kappa,n,\lambda)}$ is a unit vector pointing along the link (κ,n,λ), *i.e.*,

$$e^{(\kappa,n,\lambda)} := \frac{X(n,\lambda) - X(0,\kappa)}{|X(n,\lambda) - X(0,\kappa)|}.$$

See Fig. 6.1(a) for an illustration, or Appendix A.2 for a concrete example. In the truss model, only the components of the potential differences that are aligned with the connecting links contribute to the energy,

$$||\boldsymbol{u}||^2_{\boldsymbol{A}} = \sum_{m\in\mathbb{Z}^d}\ \sum_{(\kappa,n,\lambda)\in\mathbb{B}_+} \alpha^{(\kappa,n,\lambda)}\big|e^{(\kappa,n,\lambda)} \cdot \big(\boldsymbol{u}(m+n,\lambda) - \boldsymbol{u}(m,\kappa)\big)\big|^2.$$

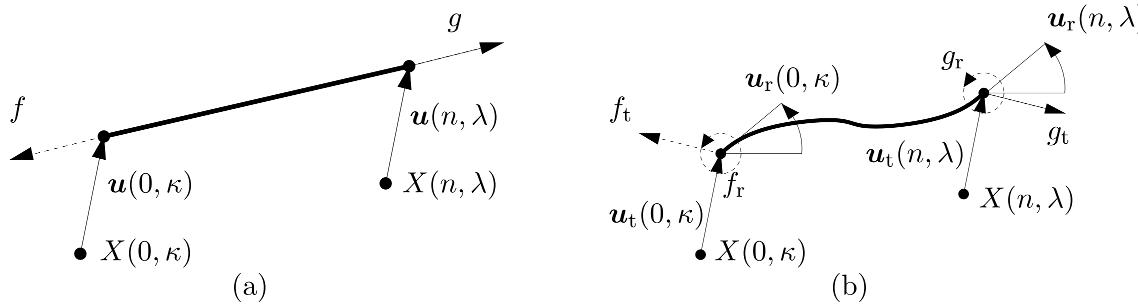

FIGURE 6.1. (a) The link connecting node $(0,\kappa)$ to (n,λ) in a truss model. While being subjected to forces f and g (drawn with dashed lines), the two ends of the link have been displaced from $X(0,\kappa)$ and $X(n,\lambda)$, to $X(0,\kappa)+\boldsymbol{u}(0,\kappa)$, and $X(n,\lambda)+\boldsymbol{u}(n,\lambda)$, respectively. (b) The links connecting node $(0,\kappa)$ to (n,λ) in a frame model. While being subjected to forces f_{t}, g_{t}, and moments f_{r}, g_{r}, the two ends of the link have been displaced from $X(0,\kappa)$ and $X(n,\lambda)$, to $X(0,\kappa)+\boldsymbol{u}_{\mathrm{t}}(0,\kappa)$, and $X(n,\lambda)+\boldsymbol{u}_{\mathrm{t}}(n,\lambda)$, respectively, and been rotated by $\boldsymbol{u}_{\mathrm{r}}(0,\kappa)$ and $\boldsymbol{u}_{\mathrm{r}}(n,\lambda)$.

The difficulty of the truss model lies in the fact that the matrix $A^{(\kappa,n,\lambda)}$ in (6.1) has rank one, which is lower than the dimensionality of the nodal potential. This is the reason that even for connected lattices such as A and D in Figure 1.1, there exist displacement fields $\boldsymbol{u}$ that are not rigid body motions but for which $||\boldsymbol{u}||_{\boldsymbol{A}}=0$.

In this model $\boldsymbol{u}(m,\kappa)\in\mathbb{C}^d$, so the symbol $\sigma(\xi)$ consists of $q\times q$ blocks, each of size $d\times d$.

6.2. **Korn's inequality.** The framework set up in Section 5 will now be used to prove the Korn inequality for truss materials. For truss lattices, the nullspace $\mathcal{N}$ is the set of rigid body motions, which has dimension $d(d+1)/2$ and for which the parameter k in property B (see Section 5) satisfies $k=1$ (meaning that a rigid body motion is uniquely determined by its restriction to Ω_1). In order to define the projection operators Ψ_{a} and Ψ_{d} (*cf.* Section 4.2) we first let I_p denote the identity matrix in $\mathbb{C}^p$, then set $\Psi_{\mathrm{a}} := q^{-1/2}[I_d,\ldots,I_d]^{\mathrm{t}}\in\mathbb{C}^{qd\times d}$ and finally choose Ψ_{d} so that $\Psi=[\Psi_{\mathrm{a}},\,\Psi_{\mathrm{d}}]$ forms a unitary $qd\times qd$ matrix. Then define $\varphi_{\mathrm{a}}=\Psi_{\mathrm{a}}^{\mathrm{t}}\varphi$ and $\varphi_{\mathrm{d}}=\Psi_{\mathrm{d}}^{\mathrm{t}}\varphi$, as before. The truss analogue of Lemma 4.1 is:

Lemma 6.1. *If a truss lattice is $\mathcal{N}$-coercive, then there exist $C<\infty$ and $c>0$ such that*

$$c\left(|\xi|^2|\varphi_{\mathrm{a}}|^2+|\varphi_{\mathrm{d}}|^2\right)\le\langle\varphi,\sigma(\xi),\varphi\rangle\le C\left(|\xi|^2|\varphi_{\mathrm{a}}|^2+|\varphi_{\mathrm{d}}|^2\right),\qquad\forall\,\varphi\in\mathbb{C}^{qd}.$$

Proof: The proof follows in part the proof of Lemma 4.1. We use Lemma 5.3, choosing $\boldsymbol{v}=\varphi_{\mathrm{a}}$, to prove the upper bound. For the lower bound, define $l_{\mathrm{a}}(\xi)$ and $l_{\mathrm{d}}(\xi)$ according to the formulae (4.5). Taking the same steps as in the proof of Lemma 4.1, we prove that $l_{\mathrm{d}}(\xi)>0$ in I^d and then compactness implies the uniform bound from below.

It remains to prove that $l_{\rm a}(\xi) \geq c|\xi|^2$. As a first step, we note that

$$\langle \varphi, \sigma(\xi), \varphi \rangle = \inf_{\boldsymbol{v} \in \mathcal{N}} \left(||\boldsymbol{u}_{\rm a} - \boldsymbol{v}_{\rm a}||_{V_2}^2 + ||\boldsymbol{u}_{\rm d} - \boldsymbol{v}_{\rm d}||_{V_2}^2 \right) \geq c \inf_{\boldsymbol{v} \in \mathcal{N}} ||\boldsymbol{u}_{\rm a} - \boldsymbol{v}_{\rm a}||_{V_2}^2. \tag{6.2}$$

It will be sufficient to keep the linear part of the map $\xi \mapsto \boldsymbol{u}(m) = \varphi e^{-\mathbf{i} m\cdot\xi}$ so we write

$$\boldsymbol{u}_{\rm a}(m) = \varphi_{\rm a} e^{-\mathbf{i} m\cdot\xi} = \varphi_{\rm a} - \varphi_{\rm a}\mathbf{i} m\cdot\xi - \varphi_{\rm a}(1 - \mathbf{i} m\cdot\xi - e^{-\mathbf{i} m\cdot\xi}),$$

and use the inequality $|a+b|^2 \geq |a|^2 - |b|^2$ to transform (6.2) into

(6.3)
$$\begin{aligned}\langle \varphi, \sigma(\xi), \varphi \rangle &\geq c \inf_{\boldsymbol{v} \in \mathcal{N}} \left(||\varphi_{\rm a} - \varphi_{\rm a}\mathbf{i}(m\cdot\xi) - \boldsymbol{v}_{\rm a}||_{V_2}^2 - ||\varphi_{\rm a}(1 - \mathbf{i} m\cdot\xi - e^{-\mathbf{i} m\cdot\xi})||_{V_2}^2 \right) \\ &\geq c \inf_{\boldsymbol{v} \in \mathcal{N}} ||\varphi_{\rm a} - \varphi_{\rm a}\mathbf{i}(m\cdot\xi) - \boldsymbol{v}_{\rm a}||_{V_2}^2 - |\varphi_{\rm a}|^2|\xi|^4.\end{aligned}$$

Let $\boldsymbol{v}_{\rm a}^{(\rm min)}$ denote the minimizer in the last term. Since $\boldsymbol{v}^{(\rm min)}$ is a rigid body motion, $\boldsymbol{v}_{\rm a}^{(\rm min)}$ can be written as a sum of a constant and a rotational component, $\boldsymbol{v}_{\rm a}^{(\rm min)} = \boldsymbol{v}_{\rm const}^{(\rm min)} + \boldsymbol{v}_{\rm rot}^{(\rm min)}$. Clearly $\boldsymbol{v}_{\rm const}^{(\rm min)} = \varphi_{\rm a}$. In order to determine $\boldsymbol{v}_{\rm rot}^{(\rm min)}$, we introduce a basis for the rotations on $\mathbb{R}^d$: For $i, j \in \{1, \dots, d\}$ such that $j - i > 0$, define the basis-vectors $\boldsymbol{w}^{(ij)} \in \mathcal{N}$ by $[\boldsymbol{w}^{(ij)}(m)]_k = \delta_{ik} m_j - \delta_{jk} m_i$. The minimizing problem can now be written

$$\inf_{\boldsymbol{v} \in \mathcal{N}} ||\varphi_{\rm a}(m\cdot\xi) - \boldsymbol{v}_{\rm a}||_{V_2}^2 = \inf_{\alpha^{(ij)}} ||\varphi_{\rm a}(m\cdot\xi) - \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} \alpha^{(ij)} \boldsymbol{w}^{(ij)}(m)||_{V_2}^2. \tag{6.4}$$

The vectors $\boldsymbol{w}^{(ij)}$ are orthogonal in V_2, so the minimizers are given by

$$\alpha^{(ij),\rm min} = \frac{\left\langle \varphi_{\rm a}(m\cdot\xi), \boldsymbol{w}^{(ij)}(m) \right\rangle_{V_2}}{||\boldsymbol{w}^{(ij)}(m)||_{V_2}^2} = \cdots = \frac{1}{2}(\varphi_{\rm A,i}\xi_j - \varphi_{\rm A,j}\xi_i).$$

We insert these values into (6.4) to obtain

$$\inf_{\alpha^{(ij)}} ||\varphi_{\rm a}(m\cdot\xi) - \sum_{i=1}^{d}\sum_{j=i+1}^{d} \alpha^{(ij)} \boldsymbol{w}^{(ij)}(m)||_{V_2}^2 = \cdots = 5^d \left(|\varphi_{\rm a}|^2|\xi|^2 + |\varphi_{\rm a}\cdot\xi|^2 \right) \geq |\varphi_{\rm a}|^2|\xi|^2.$$

Combining this expression with (6.3) we find that

$$\langle \varphi, \sigma(\xi), \varphi \rangle \geq c \left(|\varphi_{\rm a}|^2|\xi|^2 - |\varphi_{\rm a}|^2|\xi|^4 \right) \geq c|\xi|^2|\varphi_{\rm a}|^2, \tag{6.5}$$

in some neighborhood of the origin. The inequality (6.5) implies that $l_{\rm a}(\xi) \geq c|\xi|^2$. □

Introducing $L(\xi)$ as a $qd \times qd$ diagonal matrix with $|\xi|^2$ on the first d entries of the diagonal and ones on the rest, we can reformulate the statement of Lemma 6.1 compactly as $\sigma(\xi) \sim \Psi L(\xi) \Psi^{\rm t}$. Since Ψ is unitary and $\det L(\xi) = |\xi|^{2d}$, Lemma 2.1 then justifies to the following result:

Corollary 6.2. *For the truss problem on an $\mathcal{N}$-coercive lattice,* $\det \sigma(\xi) \sim |\xi|^{2d}$.

6.3. **Well posedness of the equilibrium equation.** In this section we specify under what conditions on the load $\boldsymbol{f}$, the equilibrium equation

$$\begin{cases} \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}, \\ ||\boldsymbol{u}||_{\boldsymbol{A}} < \infty, \end{cases} \tag{6.6}$$

is well-posed. As for the conduction case, this condition depends on the dimension. The proof of the following result is exactly analogous to the proof of Theorem 4.5 and is omitted.

Theorem 6.3. *(a) In two dimensions, suppose that* $|m|\boldsymbol{f}_{\mathrm{a}}(m) \in l^1$, $\sum \boldsymbol{f}(m,\kappa) = 0$, $\boldsymbol{f}_{\mathrm{d}} \in l^2$, *and that the lattice is* $\mathcal{N}$*-coercive. Then (6.6) has a solution* $\boldsymbol{u}$ *which is unique up to rigid body motions. This solution satisfies* $||\boldsymbol{u}||_{\boldsymbol{A}} \leq C\,(||m\boldsymbol{f}_{\mathrm{a}}(m)||_{l^1} + ||\boldsymbol{f}_{\mathrm{d}}||_{l^2})$.
(b) In three (and higher) dimensions, suppose that $\boldsymbol{f}_{\mathrm{a}} \in l^1$, $\boldsymbol{f}_{\mathrm{d}} \in l^2$, *and that the lattice is* $\mathcal{N}$*-coercive. Then (6.6) has a solution* $\boldsymbol{u}$ *which is unique up to rigid body motions. This solution satisfies* $||\boldsymbol{u}||_{\boldsymbol{A}} \leq C\,(||\boldsymbol{f}_{\mathrm{a}}||_{l^1} + ||\boldsymbol{f}_{\mathrm{d}}||_{l^2})$.

Remark 6.1. Theorem 6.3 is, like Theorem 4.5, a special case of a more general result in d dimensions with the requirement that $\boldsymbol{f}_{\mathrm{a}} \in S_1^2$ and that $\boldsymbol{f}_{\mathrm{d}} \in S_0^2$.

6.4. **Properties of $\sigma(\xi)^{-1}$.** In the truss model, the nodal potential is a vector and the symbol is a matrix consisting of $q \times q$ blocks, each of size $d \times d$. We can still split the symbol into components that act on the average fields, and components that act on the oscillating fields by defining $\sigma_{\alpha\beta}(\xi)$ and $\tau_{\alpha\beta}(\xi)$ through the formula (4.12), but we bear in mind that now $\sigma_{\mathrm{aa}} \in \mathbb{C}^{d\times d}$, $\sigma_{\mathrm{ad}} \in \mathbb{C}^{d\times d(q-1)}$, $\sigma_{\mathrm{da}} \in \mathbb{C}^{d(q-1)\times d)}$, and $\sigma_{\mathrm{dd}} \in \mathbb{C}^{d(q-1)\times d(q-1)}$. Lemma 6.1 then implies that

$$\sigma_{\mathrm{aa}}(\xi) \sim |\xi|^2 I_d, \qquad ||\sigma_{\mathrm{ad}}(\xi)|| = ||\sigma_{\mathrm{da}}(\xi)|| \leq C|\xi|, \qquad \sigma_{\mathrm{dd}}(\xi) \sim I_{d(q-1)}, \tag{6.7}$$

and we can therefore still define the limit symbol as

$$\sigma^{(0)}(\xi) := \left[\lim_{\varepsilon\to 0} \varepsilon^2 \tau_{\mathrm{aa}}(\varepsilon\xi)\right]^{-1} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\left[\sigma_{\mathrm{aa}}(\varepsilon\xi) - \sigma_{\mathrm{ad}}(\varepsilon\xi)\sigma_{\mathrm{dd}}(0)^{-1}\sigma_{\mathrm{da}}(\varepsilon\xi)\right], \tag{6.8}$$

noting that this is now a $d \times d$ matrix. Since $\tau_{\mathrm{aa}}(\xi) \sim |\xi|^{-2} I_d$, it must be the case that $\sigma^{(0)}(\xi) \sim |\xi|^2 I_d$. To sum up, we have proved the following:

Theorem 6.4. *Suppose that* $\sigma(\xi)$ *is the symbol of an* $\mathcal{N}$*-coercive truss lattice. To lowest order, all blocks of* $\sigma(\xi)^{-1}$ *are identical. In particular, there exists a* $d \times d$ *matrix* $\sigma^{(0)}(\xi)$, *every entry of which is a second order polynomial in* ξ, *such that* $\sigma^{(0)}(\xi) \sim |\xi|^2 I_d$ *and*

$$\sigma(\xi)^{-1} = \Psi_{\mathrm{a}} \sigma^{(0)}(\xi)^{-1} \Psi_{\mathrm{a}}^{\mathrm{t}} + O(|\xi|^{-1}), \qquad \textit{as } |\xi| \to 0.$$

Remark 6.2. In every example that we have seen, $\sigma^{(0)}(\xi)$ is the symbol of an equilibrium operator of classical elasticity. However, a general statement to this effect has not yet been proved.

7. Mechanical frames

The mechanical frame model is appropriate for skeletal structures that are connected but that would turn into mechanisms that can freely deform if considered as pin-jointed trusses. Lattices A and D in Figure 1.1 belong to this category. In the frame model, the nodes are rigid and we take the bending stiffness of the links into account. The nodal potential in this model includes both rotational and translational degrees of freedom so that in two dimensions, $\boldsymbol{u}(m,\kappa) \in \mathbb{C}^3$, in three dimensions, $\boldsymbol{u}(m,\kappa) \in \mathbb{C}^6$ and in d dimensions, $\boldsymbol{u}(m,\kappa) \in \mathbb{C}^{d(d+1)/2}$. The load $\boldsymbol{f}(m,\kappa)$ has the same dimension as $\boldsymbol{u}(m,\kappa)$ and represents both forces and torques.

The frame model will present new technical obstacles in that the equilibrium operator acts differently on the rotational and translational degrees of freedom. However, this difficulty is to some extent offset by the fact that the regularity condition on the geometry is again simply that the lattice must be connected.

7.1. **The model.** The local matrices (*cf.* (3.1)) have the general form

$$A^{(\kappa,n,\lambda)} = \begin{bmatrix} B^{(\kappa,n,\lambda)} & C^{(\kappa,n,\lambda)} \\ (C^{(\kappa,n,\lambda)})^{\mathrm{t}} & D^{(\kappa,n,\lambda)} \end{bmatrix}, \tag{7.1}$$

are of size $2r \times 2r$, and have rank r, where $r = d(d+1)/2$. Castigliano's theorem dictates that $A^{(\kappa,n,\lambda)}$ must be symmetric, but it is in general *not* the case that $C^{(\kappa,n,\lambda)} = -B^{(\kappa,n,\lambda)}$ or that $D^{(\kappa,n,\lambda)} = B^{(\kappa,n,\lambda)}$. In the frame model, the direction of a link is therefore essential (this is not the case for the truss and conduction models). An example where the struts are considered as Euler beams in two dimensions is given in Appendix A.3. For an explicit expression for $A^{(\kappa,m,\lambda)}$ in the more general case of Timoschenko beams in three dimensions, see Przemieniecki [24].

A nodal potential $\boldsymbol{u}(m,\kappa) \in \mathbb{C}^{d(d+1)/2}$ is organized so that

$$\boldsymbol{u}(m,\kappa) = \begin{bmatrix} \boldsymbol{u}_{\mathrm{t}}(m,\kappa) \\ \boldsymbol{u}_{\mathrm{r}}(m,\kappa) \end{bmatrix},$$

where $\boldsymbol{u}_{\mathrm{t}}(m,\kappa) \in \mathbb{C}^d$ represents the translational displacement and $\boldsymbol{u}_{\mathrm{r}}(m,\kappa) \in \mathbb{C}^{d(d-1)/2}$ the rotational displacement. Likewise, a load $\boldsymbol{f}(m,\kappa)$ is decomposed into a force load $\boldsymbol{f}_{\mathrm{t}}(m,\kappa) \in \mathbb{C}^d$ and a torque load $\boldsymbol{f}_{\mathrm{r}}(m,\kappa) \in \mathbb{C}^{d(d-1)/2}$, see Fig. 6.1(b). A cell-wise potential φ is decomposed analogously

$$\varphi = \begin{bmatrix} \varphi(1) \\ \vdots \\ \varphi(\kappa) \end{bmatrix} \in \mathbb{C}^{qd(d+1)/2}, \qquad \text{where} \qquad \varphi(\kappa) = \begin{bmatrix} \varphi_{\mathrm{t}}(\kappa) \\ \varphi_{\mathrm{r}}(\kappa) \end{bmatrix} \in \mathbb{C}^{d(d+1)/2}. \tag{7.2}$$

7.2. **Korn's inequality.** For the frame case, we split a vector $\varphi \in \mathbb{C}^{qd(d+1)/2}$ into an average translation $\varphi_{\mathrm{at}} \in \mathbb{C}^d$, an average rotation $\varphi_{\mathrm{ar}} \in \mathbb{C}^{d(d-1)/2}$, and an intra-cell oscillation $\varphi_{\mathrm{d}} \in \mathbb{C}^{(q-1)d(d+1)/2}$ (which contains both rotational and translational components). To change variables from the node-wise representation (7.2) to the average/difference representation we have used before, we introduce a matrix $\Psi_{\mathrm{at}} \in$

$\mathbb{C}^{qd(d+1)/2\times d}$ that computes the average translation,

$$\Psi_{\mathrm{at}}^{\mathrm{t}}\varphi = \frac{1}{\sqrt{q}}\sum_{\kappa=1}^{q}\varphi_{\mathrm{t}}(\kappa),$$

another matrix $\Psi_{\mathrm{ar}} \in \mathbb{C}^{qd(d+1)/2\times d(d-1)/2}$ that computes the average rotation,

$$\Psi_{\mathrm{ar}}\varphi = \frac{1}{\sqrt{q}}\sum_{\kappa=1}^{q}\varphi_{\mathrm{r}}(\kappa),$$

and then let $\Psi_{\mathrm{d}} \in \mathbb{C}^{qd(d+1)/2\times(q-1)d(d+1)/2}$ be such that $\Psi = [\Psi_{\mathrm{at}}\ \Psi_{\mathrm{ar}}\Psi_{\mathrm{d}}] \in \mathbb{C}^{qd(d+1)/2\times qd(d+1)/2}$ is a unitary matrix. The core coercivity result is now:

Lemma 7.1. *For a connected frame lattice, there exist $C < \infty$ and $c > 0$ such that*

$$c\left(|\xi|^2|\varphi_{\mathrm{at}}|^2 + |\varphi_{\mathrm{ar}}|^2 + |\varphi_{\mathrm{d}}|^2\right) \le \langle\varphi, \sigma(\xi), \varphi\rangle \le C\left(|\xi|^2|\varphi_{\mathrm{at}}|^2 + |\varphi_{\mathrm{ar}}|^2 + |\varphi_{\mathrm{d}}|^2\right) \qquad \forall\,\varphi \in \mathbb{C}^{qd(d+1)/2}.$$

Proof: For the frame model, the nullset $\mathcal{N}$ is again the space of rigid body motions but since we now incorporate rotational degrees of freedom in the nodal potential, a function $\boldsymbol{u} \in \mathcal{N}$ is uniquely specified by its value at a single node. In other words, using the notation of Section 5, we have $k = 0$ as in the conduction problem. Furthermore, in the frame model, the links are $\mathcal{N}$-rigid (this concept is defined at the end of Section 5) and thus $\mathcal{N}$-coercivity of the global lattice is a consequence of the connectivity requirement and Lemma 5.4.

We now proceed as before and set $\boldsymbol{u}(m,\kappa) = \varphi(\kappa)e^{-\mathbf{i}m\cdot\xi}$. In order to prove the upper bound, simply use Lemma 5.3 with $\boldsymbol{v}(m,\kappa) = [\varphi_{\mathrm{at}}^{\mathrm{t}},\,0]^{\mathrm{t}}$. For the lower bound, define

$$l_{\mathrm{at}}(\xi) = \inf_{|\varphi|=1}\frac{1}{2}\frac{\langle\varphi,\sigma(\xi),\varphi\rangle}{|\varphi_{\mathrm{at}}|^2}, \qquad \text{and} \qquad l_{\mathrm{X}}(\xi) = \inf_{|\varphi|=1}\frac{1}{2}\frac{\langle\varphi,\sigma(\xi),\varphi\rangle}{|\varphi_{\mathrm{ar}}|^2 + |\varphi_{\mathrm{d}}|^2}.$$

Since $\boldsymbol{u} \notin \mathcal{N}$ when $\xi \neq 0$, it is clear that both l_{at} and l_{X} are strictly positive for $\xi \neq 0$.

To prove that $l_{\mathrm{X}}(\xi) \ge c$, it is sufficient to prove that $l_{\mathrm{X}}(0) > 0$ (since I^d is compact). We prove this by contradiction; suppose that $l_{\mathrm{X}}(0) = 0$, then, since the infimum is taken over a compact set, there would exist a minimizer φ' such that $\langle\varphi',\sigma(0),\varphi'\rangle = 0$. By Lemma 3.3, this would mean that the function $\boldsymbol{u}'$ generated by φ' would belong to $\mathcal{N}$. Now, since $\boldsymbol{u}'$ is constant from cell to cell, it could not have a rotational component, and would have to be a pure translation. But then $\varphi'_{\mathrm{ar}} = \varphi'_{\mathrm{d}} = 0$, which shows that such a minimizer cannot exist.

Finally we need to establish that $l_{\mathrm{at}}(\xi) \ge c|\xi|^2$. Starting with the trivial inequality

$$\langle\varphi,\sigma(0),\varphi\rangle \ge c\inf_{\boldsymbol{v}\in\mathcal{N}}||\boldsymbol{u}-\boldsymbol{v}||_{V_1}^2 \ge c\inf_{\boldsymbol{v}\in\mathcal{N}}||\boldsymbol{u}_{\mathrm{at}}-\boldsymbol{v}_{\mathrm{at}}||_{V_1}^2,$$

this can be proved by taking the same steps as in the corresponding part of the proof for trusses. □

The statement of Lemma 7.1 can be rewritten as $\Psi\sigma(\xi)\Psi^{\mathrm{t}} \sim L(\xi)$, where $L(\xi) = \mathrm{diag}(|\xi|^2 I_d,\, I_{qd(d+1)/2-d})$. We note that $\det L(\xi) = |\xi|^{2d}$ and then Lemma 2.1 implies that:

Corollary 7.2. *For the frame problem on a connected lattice,* $\det\sigma(\xi) \sim |\xi|^{2d}$.

7.3. **Well posedness of the equilibrium equation.** In this section we specify under what conditions on the load $\boldsymbol{f}$, the equilibrium equation

$$\begin{cases} \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}, \\ ||\boldsymbol{u}||_{\boldsymbol{A}} < \infty, \end{cases} \tag{7.3}$$

is well-posed. As in sections 4.3 and 6.3, the conditions depend on the dimension. The proofs are exactly analogous to the corresponding proofs in Section 4.3 and are omitted.

Theorem 7.3. *(a) In two dimensions, suppose that* $|m|\boldsymbol{f}_{\mathrm{at}}(m) \in l^1$, $\sum \boldsymbol{f}(m,\kappa) = 0$, $\boldsymbol{f}_{\mathrm{ar}} \in l^2$, $\boldsymbol{f}_{\mathrm{d}} \in l^2$, *and that the lattice is connected. Then (7.3) has a solution* $\boldsymbol{u}$ *which is unique up to rigid body motions. This solution satisfies* $||\boldsymbol{u}||_{\boldsymbol{A}} \leq C\,(||m\boldsymbol{f}_{\mathrm{at}}(m)||_{l^1} + ||\boldsymbol{f}_{\mathrm{ar}}||_{l^2} + ||\boldsymbol{f}_{\mathrm{d}}||_{l^2})$.
(b) In three (and higher) dimensions, suppose that $\boldsymbol{f}_{\mathrm{at}} \in l^1$, $\boldsymbol{f}_{\mathrm{ar}} \in l^2$, $\boldsymbol{f}_{\mathrm{d}} \in l^2$, *and that the lattice is connected. Then (7.3) has a solution* $\boldsymbol{u}$ *which is unique up to rigid body motions. This solution satisfies* $||\boldsymbol{u}||_{\boldsymbol{A}} \leq C\,(||\boldsymbol{f}_{\mathrm{at}}||_{l^1} + ||\boldsymbol{f}_{\mathrm{ar}}||_{l^2} + ||\boldsymbol{f}_{\mathrm{d}}||_{l^2})$.

Remark 7.1. Theorem 7.3 is, like Theorem 4.5, a special case of a more general result in d dimensions with the requirement that $\boldsymbol{f}_{\mathrm{at}} \in S^2_1$ and that $|\boldsymbol{f}_{\mathrm{ar}}|+|\boldsymbol{f}_{\mathrm{d}}| \in S^2_0$.

7.4. **Properties of $\sigma(\xi)^{-1}$.** For frame lattices, the matrix $\sigma(\xi)^{-1}$ has a rather more complicated singular behavior than the previous models that we have studied. We define $\sigma_{\mathrm{at,at}}(\xi)$, $\sigma_{\mathrm{at,ar}}(\xi)$, *et.c.* in the natural fashion and set

$$\sigma^{(0)}(\xi) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left(\sigma_{\mathrm{at,at}}(\varepsilon\xi) - [\sigma_{\mathrm{at,ar}}(\varepsilon\xi)\ \sigma_{\mathrm{at,d}}(\varepsilon\xi)]\begin{bmatrix}\sigma_{\mathrm{ar,ar}}(0) & \sigma_{\mathrm{ar,d}}(0)\\ \sigma_{\mathrm{d,ar}}(0) & \sigma_{\mathrm{d,d}}(0)\end{bmatrix}^{-1}\begin{bmatrix}\sigma_{\mathrm{ar,at}}(\varepsilon\xi)\\ \sigma_{\mathrm{d,at}}(\varepsilon\xi)\end{bmatrix}\right) \tag{7.4}$$

which is a $d\times d$ matrix of second order polynomials. Mimicking the proof of Theorem 4.9 it is simple to prove that Lemma 7.1 implies the following result:

Theorem 7.4. *Suppose that* $\sigma(\xi)$ *is the symbol of a connected frame lattice and that* $\sigma^{(0)}(\xi)$ *is given by (7.4). Then*

$$\sigma(\xi)^{-1} = \Psi_{\mathrm{at}}\sigma^{(0)}(\xi)^{-1}\Psi_{\mathrm{at}}^{\mathrm{t}} + O(|\xi|^{-1}).$$

The matrix $\sigma^{(0)}(\xi)$ *has polynomial entries and satisfies* $\sigma^{(0)}(\xi) \sim |\xi|^2 I_d$.

Remark 7.2. The limit symbol $\sigma^{(0)}(\xi)$ corresponds to the symbol of a classical elasticity operator. This operator specifies how the translational degrees of freedom are related to the applied force field when these fields are averaged over many cells. For the more general case where torque loads are present, and the rotational displacements are sought, the corresponding relationship is a mixed order equation that corresponds to a micro-polar equation of elasticity. The details of how to derive such an equation are given [19].

7.5. **When to use the frame model.** We note that even when a lattice happens to be non-degenerate when considered as a truss, it may still be necessary to model it as a frame in order to attain a specified accuracy. In order to determine the modelling error incurred by neglecting the bending stiffness, we consider first the complete equilibrium equation in the frame model, $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$, where $\boldsymbol{u} = [\boldsymbol{u}_{\mathrm{t}}, \boldsymbol{u}_{\mathrm{r}}]$ contains both translational and rotational components and $\boldsymbol{f} = [\boldsymbol{f}_{\mathrm{t}}, \boldsymbol{f}_{\mathrm{r}}]$ both force and torque loads. We split $\boldsymbol{A}$ into different parts representing axial and bending stiffnesses, taking into account the fact that the bending stiffness is lower than the axial stiffness by a factor of β^2, where $\beta =$ diameter/length is the slenderness ratio of a typical strut,

$$\left(\begin{bmatrix} \boldsymbol{A}_{\mathrm{tt}}^{\mathrm{axial}} & 0 \\ 0 & 0 \end{bmatrix} + \beta^2 \begin{bmatrix} \boldsymbol{A}_{\mathrm{tt}}^{\mathrm{bending}} & \boldsymbol{A}_{\mathrm{tr}}^{\mathrm{bending}} \\ \boldsymbol{A}_{\mathrm{rt}}^{\mathrm{bending}} & \boldsymbol{A}_{\mathrm{rr}}^{\mathrm{bending}} \end{bmatrix}\right) \begin{bmatrix} \boldsymbol{u}_{\mathrm{t}} \\ \boldsymbol{u}_{\mathrm{r}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_{\mathrm{t}} \\ \boldsymbol{f}_{\mathrm{r}} \end{bmatrix}. \tag{7.5}$$

The operator $\boldsymbol{A}_{\mathrm{tt}}^{\mathrm{axial}}$ is the stiffness operator in the truss model which we assume to be invertible (if it is not, the frame model *must* be used). If there is no torque load, we can eliminate $\boldsymbol{u}_{\mathrm{r}}$ whence

(7.6)
$$\boldsymbol{u}_{\mathrm{t}} = \left[\boldsymbol{A}_{\mathrm{tt}}^{\mathrm{axial}} + \beta^2 \boldsymbol{A}_{\mathrm{tt}}^{\mathrm{bending}} - \beta^2 \boldsymbol{A}_{\mathrm{tr}}^{\mathrm{bending}} (\boldsymbol{A}_{\mathrm{rr}}^{\mathrm{bending}})^{-1} \boldsymbol{A}_{\mathrm{rt}}^{\mathrm{bending}}\right]^{-1} \boldsymbol{f}_{\mathrm{t}} = \left[\boldsymbol{A}_{\mathrm{tt}}^{\mathrm{axial}}\right]^{-1} \boldsymbol{f}_{\mathrm{t}} + O(\beta^2).$$

Since β is small, it seems safe to drop the $O(\beta^2)$-term and model the material as a pin-jointed truss. However, to ascertain that this is truly justified, one should verify that the smallest eigenvalue (as opposed to the individual elements) of $\boldsymbol{A}_{\mathrm{tt}}^{\mathrm{axial}}$ is sufficiently much larger than β^2. This can easily be done in the Fourier setting since there, the relevant operators all have compact support.

8. Summary

We have investigated the equations associated with three different physical models: thermostatic equilibrium on lattices and elastostatic equilibrium on mechanical truss and frame lattices. The three models are summarized in Table 1. For each model, we proved that a version of Korn's inequality holds for the case of an infinite periodic domain provided that the micro-structure satisfies certain geometric constraints (for conduction and mechanical frame problems, the constraint is proven minimal). We used this inequality to formulate conditions under which the lattice equilibrium equation is well-posed for each of the three models and described the nature of the inverse operator.

The results presented are essential in the design of efficient numerical methods, see [18]. They can also be used to construct a fundamental solution to the equilibrium equation, which can then be used to study equations on finite domains by means of discrete boundary equations, see [21] and [25]. Finally, they are fundamental to an analysis of the asymptotic behavior of the solution of the lattice equations as the lattice cell size tends to zero, see [19].

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Model:	Conduction	Mechanical truss	Mechanical frame
Potential:	Temperature	Displacement	Displacement and rotation
Load:	Heat source/sink	Force	Force and torque
Dim. of potential:	1	d	$d(d+1)/2$
Rank($A^{(\kappa,n,\lambda)}$)	1	1	$d(d+1)/2$
Nullspace $\mathcal{N}$:	Constants	Rigid body motions	Rigid body motions
Non-deg. condition:	Connectivity	$\mathcal{N}$-coercivity	Connectivity

TABLE 1. Different lattice models; d is the dimension of the surrounding space and $A^{(\kappa,n,\lambda)}$ is the local stiffness matrix.

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Appendix A. Model problems

A.1. **Conduction in a multi-atomic lattice.** Consider heat conduction on the square lattice with X-braces labelled B in Figure 1.1. This lattice has two types of nodes, those of "type 1" that connect to eight links, and those of "type 2" that connect to four. Let $\boldsymbol{u}(m,1)$ and $\boldsymbol{u}(m,2)$ denote the temperatures of the respective nodes in cell $m \in \mathbb{Z}^2$ and let $\boldsymbol{f}(m,1)$ and $\boldsymbol{f}(m,2)$ denote external heat sources (so that $\boldsymbol{u}(m) = [\boldsymbol{u}(m,1),\, \boldsymbol{u}(m,2)]^{\mathrm{t}}$ and $\boldsymbol{f}(m) = [\boldsymbol{f}(m,1),\, \boldsymbol{f}(m,2)]^{\mathrm{t}}$). When the horizontal and vertical links have conductivity 1, and the diagonal ones conductivity α, the equilibrium equations read

$$\begin{aligned}
\boldsymbol{f}(m,1) = {} & \big[4\boldsymbol{u}(m,1) - \boldsymbol{u}(m-e_1,1) - \boldsymbol{u}(m+e_1,1) - \boldsymbol{u}(m-e_2,1) - \boldsymbol{u}(m+e_2,1)\big] + \\
& \alpha\big[4\boldsymbol{u}(m,1) - \boldsymbol{u}(m,2) - \boldsymbol{u}(m-e_1,2) - \boldsymbol{u}(m-e_2,2) - \boldsymbol{u}(m-e_1-e_2,2)\big], \\
\boldsymbol{f}(m,2) = {} & \alpha\big[4\boldsymbol{u}(m,2) - \boldsymbol{u}(m,1) - \boldsymbol{u}(m+e_1,1) - \boldsymbol{u}(m+e_2,1) - \boldsymbol{u}(m+e_1+e_2,1)\big],
\end{aligned}$$

where $e_1 = [1,0]^{\mathrm{t}}$ and $e_2 = [0,1]^{\mathrm{t}}$. The symbol of this system is

$$\sigma(\xi) = \begin{bmatrix} 4 - e^{\mathbf{i}\xi_1} - e^{-\mathbf{i}\xi_1} - e^{\mathbf{i}\xi_2} - e^{-\mathbf{i}\xi_2} + 4\alpha & -\alpha(1 + e^{\mathbf{i}\xi_1} + e^{\mathbf{i}\xi_2} + e^{\mathbf{i}(\xi_1+\xi_2)}) \\ -\alpha(1 + e^{-\mathbf{i}\xi_1} + e^{-\mathbf{i}\xi_2} + e^{-\mathbf{i}(\xi_1+\xi_2)}) & 4\alpha \end{bmatrix}.$$

Using the relation $-e^{\mathbf{i}\theta} + 2 - e^{-\mathbf{i}\theta} = 4\sin^2(\theta/2)$ we find that

$$\begin{aligned}
\det\sigma(\xi) = 16\alpha\left(1 + \frac{\alpha}{2}\right)\left(\sin^2\frac{\xi_1}{2} + \sin^2\frac{\xi_2}{2}\right) + 4\alpha^2\left(\sin^2\frac{\xi_1+\xi_2}{2} + \sin^2\frac{\xi_1-\xi_2}{2}\right) \\
= 4\alpha(1+\alpha)|\xi|^2 + O(|\xi|^4).
\end{aligned}$$

The inverse symbol is then given by

$$[\sigma(\xi)]^{-1} = \frac{1}{\det\sigma(\xi)}\begin{bmatrix} 4\alpha & \alpha\left(1 + e^{\mathbf{i}\xi_1} + e^{\mathbf{i}\xi_2} + e^{\mathbf{i}(\xi_1+\xi_2)}\right) \\ \alpha\left(1 + e^{-\mathbf{i}\xi_1} + e^{-\mathbf{i}\xi_2} + e^{-\mathbf{i}(\xi_1+\xi_2)}\right) & 4\sin^2\frac{\xi_1}{2} + 4\sin^2\frac{\xi_2}{2} + 4\alpha \end{bmatrix}.$$

Since $\det \sigma(\xi) \geq c|\xi|^2$ for some constant $c > 0$, the inverse exists for any $\xi \neq 0$, and near the origin the inverse symbol has the singularity

$$\sigma(\xi)^{-1} = \frac{1}{(1+\alpha)|\xi|^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2(1+\alpha)|\xi|^2} \begin{bmatrix} 0 & \mathbf{i}\xi_1 + \mathbf{i}\xi_2 \\ -\mathbf{i}\xi_1 - \mathbf{i}\xi_2 & 0 \end{bmatrix} + O(1),$$

where $O(1)$ stands for a function of ξ that remains uniformly bounded as $|\xi| \to 0$.

A.2. **A multi-atomic truss lattice.** We consider the same lattice as in Section A.2 but now we model it as a mechanical truss. We give the horizontal and vertical bars stiffness 1, the diagonal ones stiffness α and set $e_1 = [1,0]^{\mathrm{t}}$, $e_2 = [0,1]^{\mathrm{t}}$, $e_3 = (1/\sqrt{2})[1,1]^{\mathrm{t}}$ and $e_4 = (1/\sqrt{2})[1,-1]^{\mathrm{t}}$. Then the equilibrium equation reads, *cf.* (6.1),

$$\begin{aligned} \boldsymbol{f}(m,1) = {} & e_1 e_1^{\mathrm{t}}\big(-\boldsymbol{u}(m-e_1,1) + 2\boldsymbol{u}(m,1) - \boldsymbol{u}(m+e_1,1)\big) + \\ & e_2 e_2^{\mathrm{t}}\big(-\boldsymbol{u}(m-e_2,1) + 2\boldsymbol{u}(m,1) - \boldsymbol{u}(m+e_2,1)\big) + \\ & e_3 e_3^{\mathrm{t}}\big(-\boldsymbol{u}(m-e_1-e_2,2) + 2\boldsymbol{u}(m,1) - \boldsymbol{u}(m,2)\big) + \\ & e_4 e_4^{\mathrm{t}}\big(-\boldsymbol{u}(m+e_1-e_2,2) + 2\boldsymbol{u}(m,1) - \boldsymbol{u}(m-e_1+e_2,2)\big), \\ \boldsymbol{f}(m,2) = {} & e_3 e_3^{\mathrm{t}}\big(-\boldsymbol{u}(m,1) + 2\boldsymbol{u}(m,2) - \boldsymbol{u}(m+e_1+e_2,1)\big) + \\ & e_4 e_4^{\mathrm{t}}\big(-\boldsymbol{u}(m+e_1,1) + 2\boldsymbol{u}(m,2) - \boldsymbol{u}(m+e_2,1)\big). \end{aligned}$$

The symbol is now given by

$$\sigma(\xi) = \begin{bmatrix} [\sigma(\xi)]_{11} & [\sigma(\xi)]_{12} \\ [\sigma(\xi)]_{12}^{\mathrm{t}} & [\sigma(\xi)]_{22} \end{bmatrix}$$

where

$$\begin{aligned} [\sigma(\xi)]_{11} &= \begin{bmatrix} 4\sin^2\frac{\xi_1}{2} + 2\alpha & 0 \\ 0 & 4\sin^2\frac{\xi_1}{2} + 2\alpha \end{bmatrix}, \\ [\sigma(\xi)]_{12} &= -\frac{\alpha}{2} \begin{bmatrix} 1 + e^{\mathbf{i}\xi_1} + e^{\mathbf{i}\xi_2} + e^{\mathbf{i}(\xi_1+\xi_2)} & 1 - e^{\mathbf{i}\xi_1} - e^{\mathbf{i}\xi_2} + e^{\mathbf{i}(\xi_1+\xi_2)} \\ 1 - e^{\mathbf{i}\xi_1} - e^{\mathbf{i}\xi_2} + e^{\mathbf{i}(\xi_1+\xi_2)} & 1 + e^{\mathbf{i}\xi_1} + e^{\mathbf{i}\xi_2} + e^{\mathbf{i}(\xi_1+\xi_2)} \end{bmatrix}, \\ [\sigma(\xi)]_{22} &= \begin{bmatrix} 2\alpha & 0 \\ 0 & 2\alpha \end{bmatrix}. \end{aligned}$$

Computing $\sigma^{(0)}(\xi)$ using (6.8) we find that

$$\sigma^{(0)}(\xi) = \begin{bmatrix} (1+\alpha/2)\xi_1^2 + (\alpha/2)\xi_2^2 & \alpha\xi_1\xi_2 \\ \alpha\xi_1\xi_2 & (\alpha/2)\xi_1^2 + (1+\alpha/2)\xi_2^2 \end{bmatrix}.$$

The lowest order system of homogenized equations is thus

$$\begin{aligned} -\big((1+\alpha/2)\partial_1^2 + (\alpha/2)\partial_2^2\big)u_1 - (\alpha\partial_1\partial_2)u_2 &= f_1, \\ -(\alpha\partial_1\partial_2)u_1 - \big((\alpha/2)\partial_1^2 + (1+\alpha/2)\partial_2^2\big)u_1 &= f_2. \end{aligned}$$

These are equations of two-dimensional elasticity for any $\alpha > 0$. If $\alpha = 1$, we get the equations of an isotropic medium (in plane stress) with Young's modulus 4/3 and Poisson's ratio 1/3. Note that if we model an actual physical truss, the isotropic case corresponds to one where the diagonal bars have a cross-sectional area that is

$\sqrt{2}$ smaller than the horizontal and vertical ones (since axial stiffness scales as area divided by length).

A.3. **A mono-atomic frame lattice.** We consider the square lattice labelled A in Figure 1.1. Modelling the struts as Euler beams with Young's modulus E, cross-sectional area A, length L and moment of intertia I, the stiffness matrix for a horizontal link is, *cf.* (7.1),

$$A^{(1,[1,0],1)} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^3} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^3} & \frac{4EI}{L} \end{bmatrix}. \tag{A.1}$$

The symbol for this lattice is then

$$\sigma(\xi) = \begin{bmatrix} \frac{EA}{L} 4\sin^2\frac{\xi_1}{2} + \frac{12EI}{L^3} 4\sin^2\frac{\xi_2}{2} & 0 & \frac{12EI}{L^2}\mathbf{i}\sin\xi_2 \\ 0 & \frac{EA}{L} 4\sin^2\frac{\xi_2}{2} + \frac{12EI}{L^3} 4\sin^2\frac{\xi_1}{2} & -\frac{12EI}{L^2}\mathbf{i}\sin\xi_1 \\ -\frac{12EI}{L^2}\mathbf{i}\sin\xi_2 & \frac{12EI}{L^2}\mathbf{i}\sin\xi_1 & \frac{4EI}{L}(4+\cos\xi_1+\cos\xi_2) \end{bmatrix},$$

and the limit symbol, *cf.* (7.4),

$$\sigma^{(0)}(\xi) = \begin{bmatrix} \frac{EA}{L}\xi_1^2 + \frac{6EI}{L^3}\xi_2^2 & \frac{6EI}{L^3}\xi_1\xi_2 \\ \frac{6EI}{L^3}\xi_1\xi_2 & \frac{EA}{L}\xi_2^2 + \frac{6EI}{L^3}\xi_2^1 \end{bmatrix}.$$

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