

Sec 2.6: Systems of differential equations

Consider the differential equation

$$(1) \quad \begin{cases} dx/dt = f(t, x, y) \\ dy/dt = g(t, x, y), \end{cases}$$

for the two unknown functions $x = x(t)$ and $y = y(t)$.

We call an equation such as (1) a *system of differential equations*.

Special case: We say that the system is *uncoupled* if

$$\begin{cases} dx/dt = f(t, x) \\ dy/dt = g(t, y), \end{cases}$$

In this case, there really just are two independent equations — you can solve each one individually without caring about the other one.

Example: Consider the equation

$$\begin{cases} dx/dt = 2x \\ dy/dt = -3y. \end{cases}$$

This is uncoupled. The solution is: $x(t) = A e^{2t}$ and $y(t) = B e^{-3t}$.

Sec 2.6: Systems of differential equations

Consider the differential equation

$$(1) \quad \begin{cases} dx/dt = f(t, x, y) \\ dy/dt = g(t, x, y), \end{cases}$$

for the two unknown functions $x = x(t)$ and $y = y(t)$.

We call an equation such as (1) a *system of differential equations*.

Special case: We say that the system is *autonomous* if

$$(2) \quad \begin{cases} dx/dt = f(x, y) \\ dy/dt = g(x, y), \end{cases}$$

In this case, the equation itself does not change with time.

Example:

$$\begin{cases} dx/dt = x + t y, \\ dy/dt = \sin(y - x). \end{cases}$$

This equation is *coupled* and *non-autonomous*.

Example:

$$\begin{cases} dx/dt = \sin(t + x), \\ dy/dt = t \cos(y). \end{cases}$$

This equation is *un-coupled* and *non-autonomous*.

Example:

$$\begin{cases} dx/dt = x + y, \\ dy/dt = \sin(y). \end{cases}$$

This equation is *coupled* and *autonomous*.

But: You can solve the y equation first, and then solve for x , so they are not coupled in any meaningful sense.

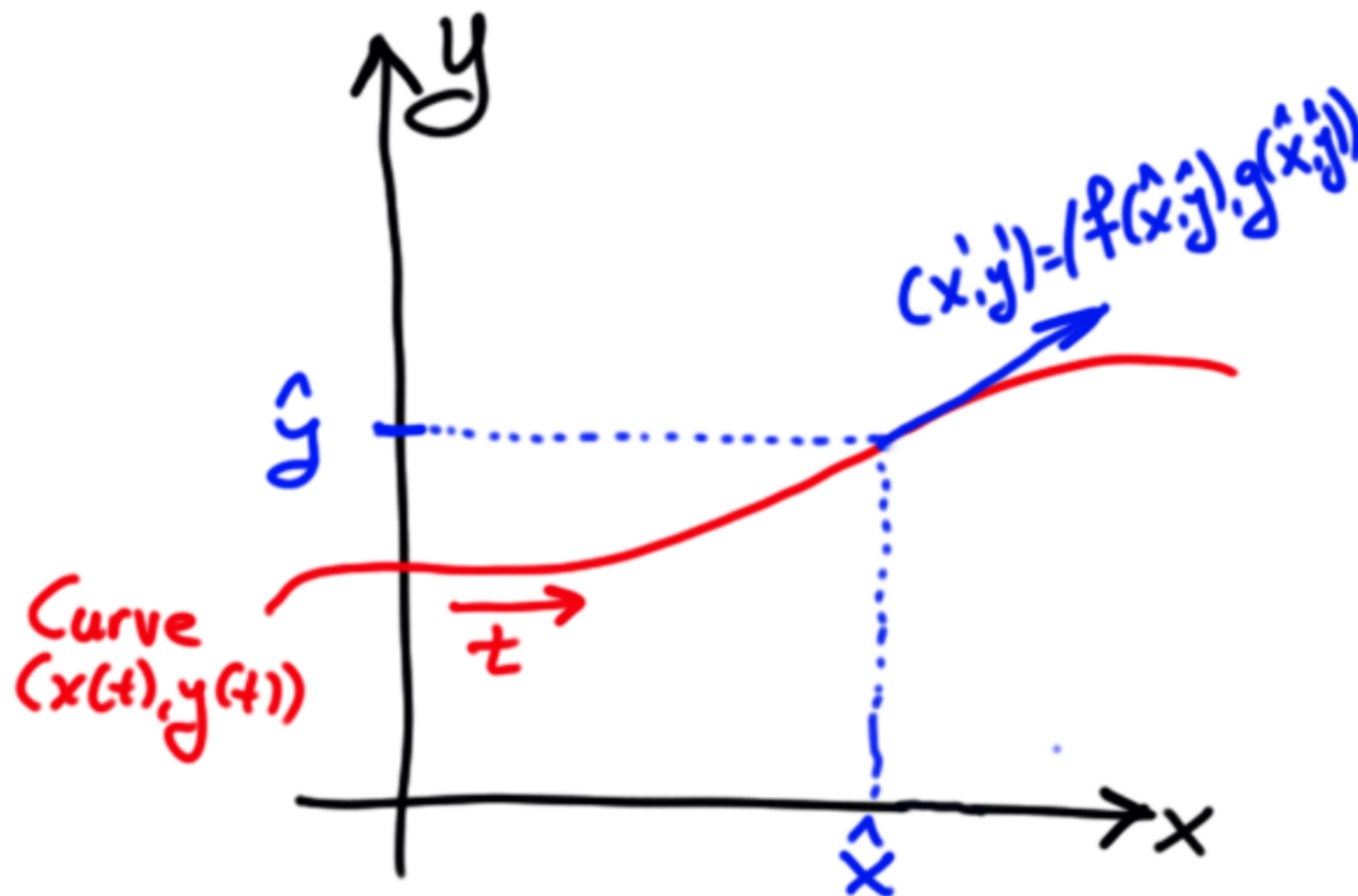
Phase diagrams: Consider an *autonomous* system $\begin{cases} dx/dt = f(x, y), \\ dy/dt = g(x, y). \end{cases}$

We will describe a *qualitative* technique for investigating the behavior of solutions.

Suppose $(x(t), y(t))$ is a solution. Observe that $\mathbf{r}(t) = (x(t), y(t))$ is a curve in the x - y -plane. Suppose this curve passes through a point (\hat{x}, \hat{y}) . Then the *slope* of the curve at this point is

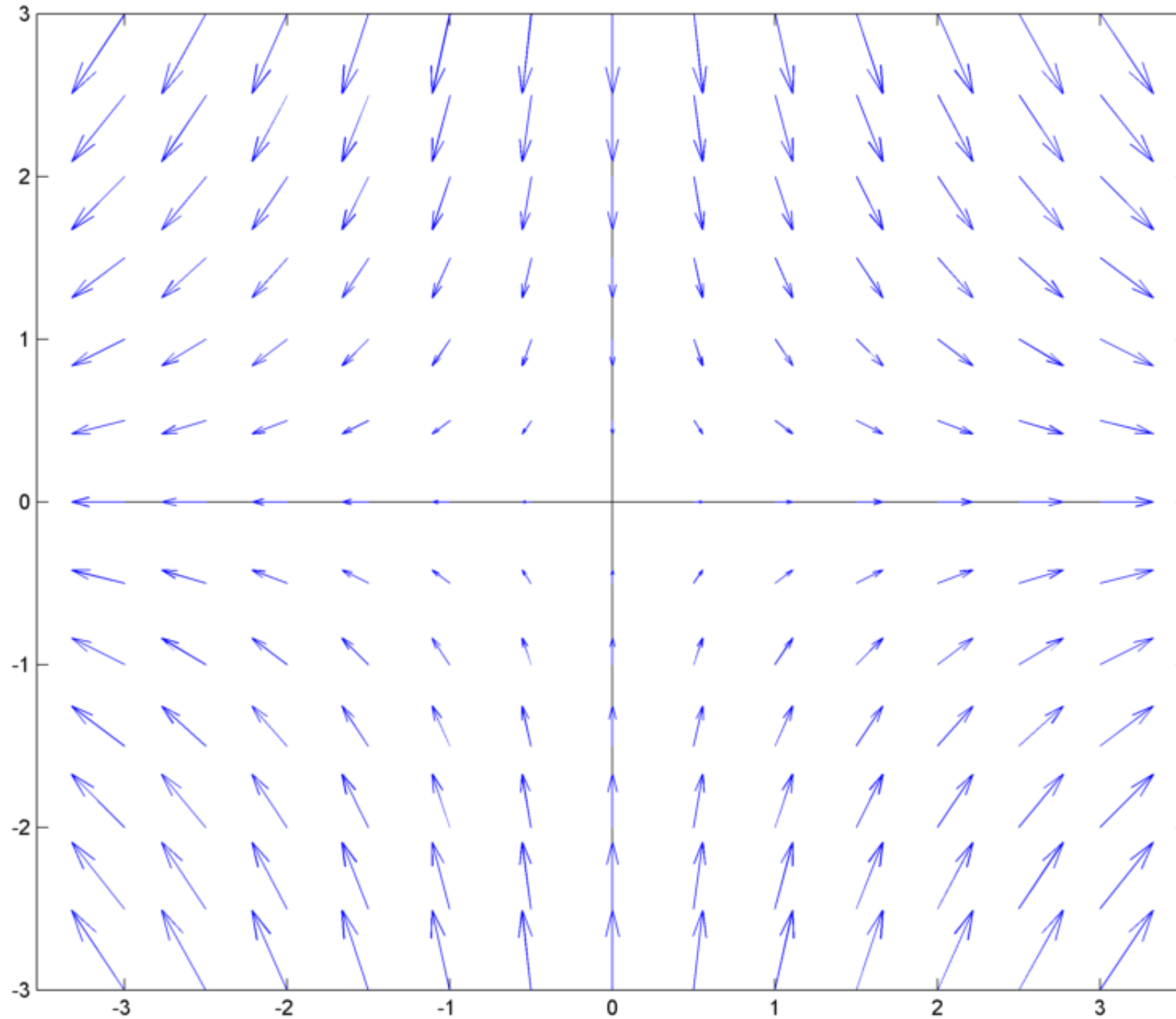
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(\hat{x}, \hat{y})}{f(\hat{x}, \hat{y})}.$$

The point is that you can tell the slope without solving the equation!



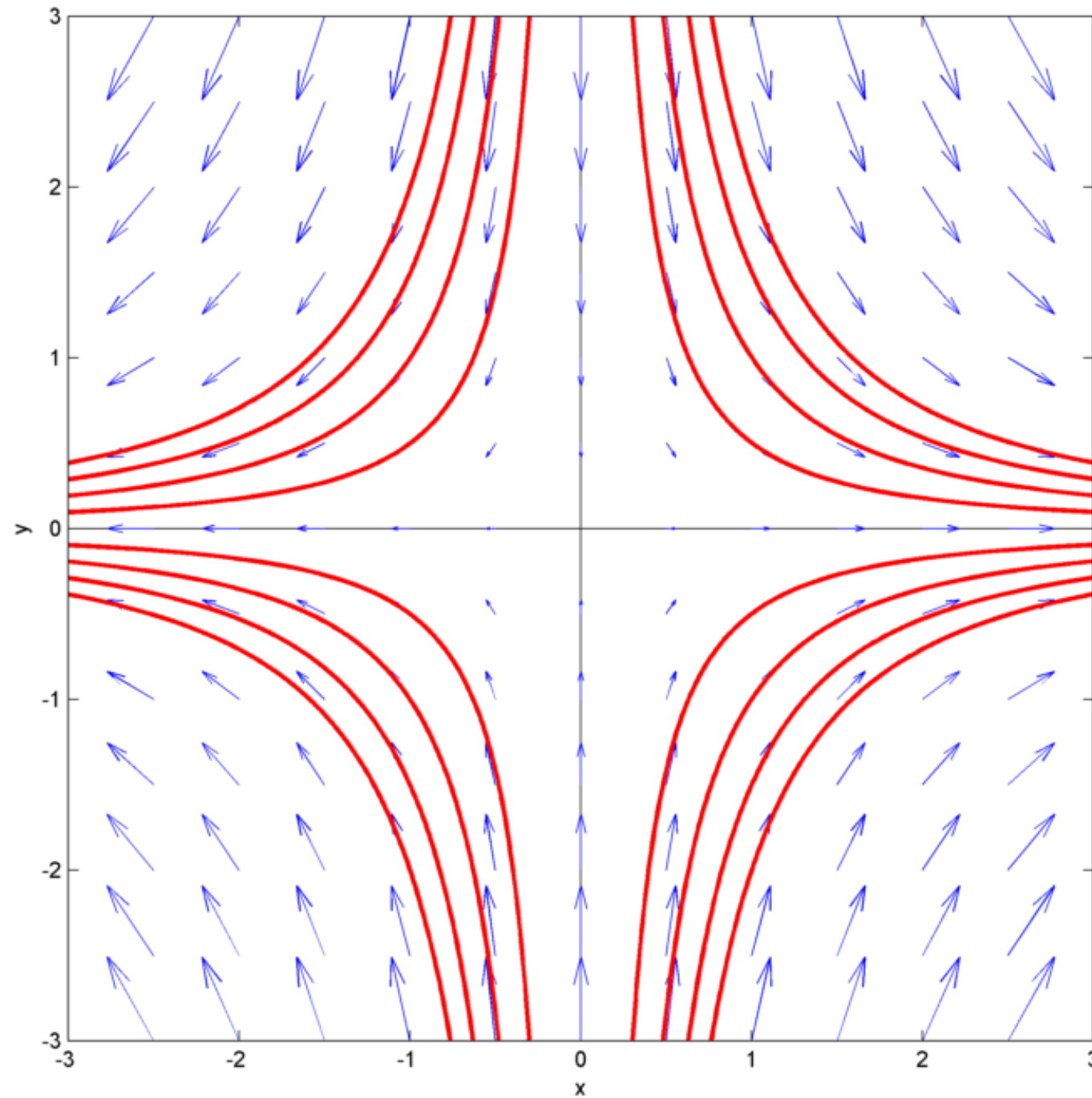
Example 1: Consider the (uncoupled, autonomous) system

$$\begin{cases} dx/dt = 2x, \\ dy/dt = -3y. \end{cases}$$



Example 1: Consider the (uncoupled, autonomous) system

$$\begin{cases} dx/dt = 2x, \\ dy/dt = -3y. \end{cases}$$



Example 1: Consider the (uncoupled, autonomous) system

$$\begin{cases} dx/dt = 2x, \\ dy/dt = -3y. \end{cases}$$

Since this is a simple uncoupled equation, we can actually solve it analytically:

$$\begin{aligned} x(t) &= A e^{2t}, \\ y(t) &= B e^{-3t}. \end{aligned}$$

Once we have the analytical solution $\mathbf{r}(t) = (x(t), y(t))$, we can also find the solution curves $y = y(x)$. First note that

$$e^t = |x(t)/A|^{1/2}.$$

Then we find that

$$y = B e^{-3t} = B (e^t)^{-3} = B (|x(t)|^{1/2}/|A|^{1/2})^{-3} = B |A|^{-3/2} |x|^{-3/2} = C |x|^{-3/2}.$$

Consider a system of differential equations

$$\begin{cases} dx/dt = f(x, y), \\ dy/dt = g(x, y). \end{cases}$$

Definition: An *equilibrium solution* is a constant solution

$$\begin{aligned} x(t) &= \hat{x}, \\ y(t) &= \hat{y}, \end{aligned}$$

for some numbers \hat{x} and \hat{y} . To find an equilibrium point (\hat{x}, \hat{y}) , solve

$$\begin{aligned} f(\hat{x}, \hat{y}) &= 0, \\ g(\hat{x}, \hat{y}) &= 0. \end{aligned}$$

Stable equilibrium point: **Every solution** that starts “near” the point enters the point.

Unstable equilibrium point: **Some solution** that starts “near” the point leaves the point.

(A more precise definition will appear later in the course.)

Example 2 (predator-prey): Consider an island on which we have

$R(t)$ = number of rabbits at time t ,

$F(t)$ = number of foxes at time t .

At time $t = 0$, there are R_0 rabbits and F_0 foxes.

Suppose first there is a fence between the rabbits and the foxes, that there is infinitely much rabbit food, and that foxes can only eat rabbits. Then the rabbit population will grow exponentially, and the fox population will decay exponentially

$$\begin{cases} R'(t) = aR, \\ F'(t) = -cF. \end{cases}$$

Now remove the fence. Then the new equation is:

$$\begin{cases} R'(t) = aR(1 - bF), \\ F'(t) = -cF(1 - dR). \end{cases}$$

The equilibrium solutions are:

$$\begin{aligned} (\hat{R}, \hat{F}) &= (0, 0) && \text{No rabbits and no foxes.} \\ (\hat{R}, \hat{F}) &= (1/d, 1/b) && \text{Perfect equilibrium.} \end{aligned}$$

Example 3 (food competition): Consider an island on which we have

$R(t)$ = number of rabbits at time t ,

$S(t)$ = number of sheep at time t .

At time $t = 0$, there are R_0 rabbits and S_0 sheep. Suppose first the rabbits and the sheep eat *different food*, so that each population is governed by its own logistic equation:

$$\begin{cases} R'(t) = aR(1 - R/L_r), \\ S'(t) = bS(1 - S/L_s). \end{cases}$$

Next suppose that the two species compete for food:

$$\begin{cases} R'(t) = aR(1 - R/L_r - cS), \\ S'(t) = bS(1 - S/L_s - dR). \end{cases}$$

The equilibrium solutions are:

- $(\hat{R}, \hat{S}) = (0, 0)$ No rabbits and no sheep.
- $(\hat{R}, \hat{S}) = (L_r, 0)$ Rabbits at carrying capacity, sheep extinct.
- $(\hat{R}, \hat{S}) = (0, L_s)$ Rabbits extinct, sheep at carrying capacity.
- $(\hat{R}, \hat{S}) = (R_*, S_*)$ Perfect equilibrium.

The point (R_*, S_*) satisfies

$$1 = R_*/L_r + cS_*, \quad 1 = S_*/L_s + dR_*.$$