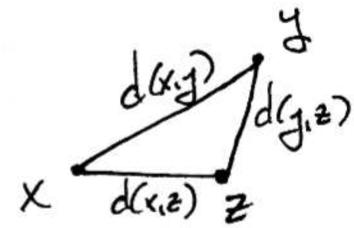
Metric Spaces

Def^n Let $\mathcal{X}$ be a non-empty set, and suppose that d is a function

$$d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$$

such that



(i) $d(x,y) = 0 \iff x = y$

(ii) $d(x,y) = d(y,x) \quad \forall x, y \in \mathcal{X}$

(iii) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x, y, z \in \mathcal{X}$

Then $(\mathcal{X}, d)$ is called a METRIC SPACE with the METRIC d.

Example $\mathbb{R}^n$ with $d(x,y) = \|x - y\|$

Example $\mathbb{R}^n$ with $d(x,y) = \max\limits_{1 \leq j \leq n} |x_j - y_j| \qquad x = (x_1, x_2, \dots, x_n)$

Proof (i) & (ii) are obvious.

For (iii) we find

$$d(x,y) = \max_{1 \leq j \leq n} |x_j - y_j| = \max |(x_j - z_j) + (z_j - y_j)| \leq$$

$$\leq \max\left(|x_j - z_j| + |z_j - y_j|\right) \leq \max |x_j - z_j| + \max |z_j - y_j|$$

$$= d(x,z) + d(z,y)$$

Example Let $\mathcal{X}$ be any set.

Define $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

<u>Claim</u> Let (X, d) be a metric space.
Let Y be a subset of X.
Then (Y, d) is a metric space.
We call (Y, d) a <u>subspace</u> of (X, d).

<u>Example</u> $X = \mathbb{R}$ $\quad d(x,y) = |x-y|$
Then $(\mathbb{Q}, d)$ is a subspace of $(\mathbb{R}, d)$

<u>Def^n</u> Let F be a scalar field ($\mathbb{R}$ or $\mathbb{C}$).
Then a set X is called a VECTOR FIELD over F (or a LINEAR SPACE) if there exist operations "$+$" and "$\cdot$" such that

X is a commutative group w.r.t. $+$:

(i) $x+y = y+x \quad \forall x,y \in X$

(ii) $(x+y)+z = x+(y+z) \quad \forall x,y,z \in X$

(iii) $\exists\, 0 \in X$ such that $x+0 = x \quad \forall x \in X$

(iv) $\forall\, x \in X \quad \exists$ an element "$-x$" $\in X$ s.t. $x+(-x)=0$

Conditions on scalar multiplication:

(v) $1 \cdot x = x \quad \forall x \in X$

(vi) $(\lambda+\mu)x = \lambda x + \mu x \quad \forall \lambda, \mu \in F \quad x \in X$

(vii) $\lambda(\mu x) = (\lambda\mu)x$ $\quad$ —— " ——

(viii) $\lambda(x+y) = \lambda x + \lambda y \quad \forall \lambda \in F \quad x, y \in X$

Def^n A linear space $\mathfrak{X}$ (over F) is called a NORMED LINEAR SPACE if there exists a map $\|\cdot\| : \mathfrak{X} \to [0,\infty)$ such that

(i) $\|\lambda x\| = |\lambda| \, \|x\|$

(ii) $\|x+y\| \leq \|x\| + \|y\|$

(iii) $\|x\| = 0 \iff x = 0$

Claim A normed linear space is a metric space with the metric $d(x,y) = \|x-y\|$.

Example $\mathfrak{X} = \mathbb{R}^n$ $\quad \|x\|_p = \left(\sum_{j=1}^{d} |x_j|^p \right)^{1/p}$ for ~~⇒~~ $p<\infty$

$\|x\|_\infty = \cancel{\sup} \max_{1 \leq j \leq n} |x_j| \; \left(= \lim_{p\to\infty} \|x\|_p\right)$

Claim $(\mathfrak{X}, \|\cdot\|_p)$ is a NLS when $1 \leq p \leq \infty$

Proof (i) & (iii) are trivial

(ii) is a little bit of work unless $p=1,2,\infty$

Homework!

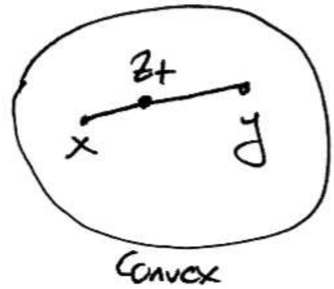
What about the case $p<1$? Let's take a detour:

Def^n Let A be a subset of a NLS $\mathfrak{X}$.

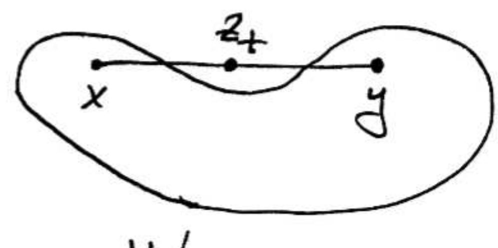
We say that A is CONVEX if $\forall x,y \in A$ & $t \in [0,1]$,

$z_t = tx + (1-t)y \in A$

Examples



Convex



Not convex

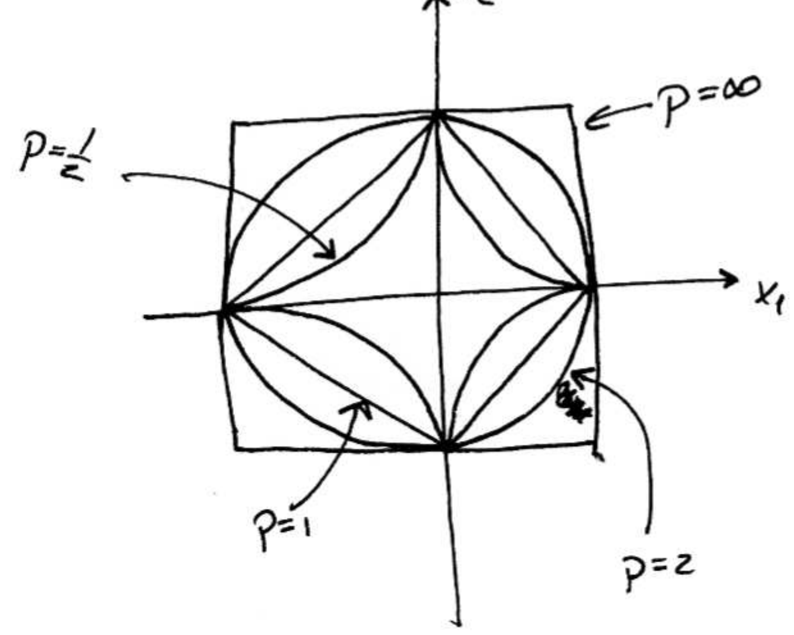
Lemma Let X be a NLS and set
$B = \{x \in X : \|x\| \leq 1\}$ ← "Closed unit ball"
Then B is convex

Proof Pick $x, y \in B$ and $t \in [0,1]$. Then
$\|z_t\| = \|tx + (1-t)y\| \leq \|tx\| + \|(1-t)y\| = t\|x\| + (1-t)\|y\| \leq 1$
So $z_t \in B$.

Claim $(\mathbb{R}^n, \|\cdot\|_p)$ is NOT a NLS ~~was~~ when $p<1$.

Proof We will prove that the unit ball is not convex.
Set $e^{(1)} = [1, 0, \ldots, 0]$, $e^{(2)} = [0, 1, 0, \ldots, 0]$, then $e^{(1)}$ & $e^{(2)} \in B$.
$z_{1/2} = [\frac{1}{2}, \frac{1}{2}, 0, 0, \ldots, 0]$.
$\|z_{1/2}\|_p = \left(\frac{1}{2^p} + \frac{1}{2^p}\right)^{1/p} = \left(2\, 2^{-p}\right)^{1/p} = 2^{\frac{1}{p}-1} > 1$ if $p<1$
so $z_{1/2} \notin B$.

In $\mathbb{R}^2$, we find that the unit ball has the following shapes:



CONVERGENCE, COMPLETENESS

<u>Def^n</u> Let $(\mathfrak{X}, d)$ be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathfrak{X}$.

* We say that $x_n \to x$ if $d(x_n, x) \to 0$. CONVERGENCE
* We say that (x_n) is CAUCHY if $\forall \varepsilon > 0$ $\exists N$ such that $m, n \geq N \Rightarrow d(x_n, x_m) < \varepsilon$
* We say that $\mathfrak{X}$ is COMPLETE if every Cauchy sequence in $\mathfrak{X}$ has a limit point in $\mathfrak{X}$.

<u>Note:</u> Every convergent sequence is necessarily Cauchy.

<u>Thm</u>: $\mathbb{R}^n$ is a complete metric space. ← Important.

<u>Example</u> $\mathfrak{X} = (0,1)$. Is $\mathfrak{X}$ complete? No: $x_n = \frac{1}{n}$ is Cauchy.

<u>Example</u> Every closed subset of $\mathbb{R}^n$ is a complete metric space. (We will return to the ~~d~~ question of when a set is "closed" shortly.)

<u>Example</u> $\mathbb{Q}$ with the usual metric is not complete. For a counterexample, pick for any integer n a rational ~~numb~~ number $q_n \in (\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n})$ Then $(q_n)_{n=1}^{\infty}$ is Cauchy, but ~~not~~ does not converge to a point in $\mathbb{Q}$.

UPPER & LOWER BOUNDS

Def^n Let A be a subset of $\mathbb{R}$.

* If M is a number such that $x \leq M \ \forall x \in A$, then M is an UPPER BOUND of A.
* If M is an upper bound of A, and there is no smaller upper bound then we say that M is the *LEAST UPPER BOUND of A, $M = \sup A$
* If A does not have an upper bound, set $\sup A = \infty$
* Analogously, define LOWER BOUND, GREATEST LOWER BOUND, $\inf A$
* For the empty set, define $\sup \emptyset = -\infty$
* If $M = \sup A \in A$, set $M = \max A$

Thm Every subset of the real numbers has a unique supremum and infimum in the reals.

Note that this is not true for max/min.
Nor is it true for subsets of $\mathbb{Q}$, example: $\Omega = \{q \in \mathbb{Q} : q^2 < 2\}$

Def^n A sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ is MONOTONE INCREASING if $x_n \leq x_{n+1} \ \forall n$
—— " —— DECREASING if $x_n \geq x_{n+1} \ \forall n$

Every monotone sequence "converges" (possibly to $\pm\infty$),
if (x_n) is monotone increasing, then $\lim_{n\to\infty} x_n = \sup\{x_n\}_{n=1}^{\infty}$

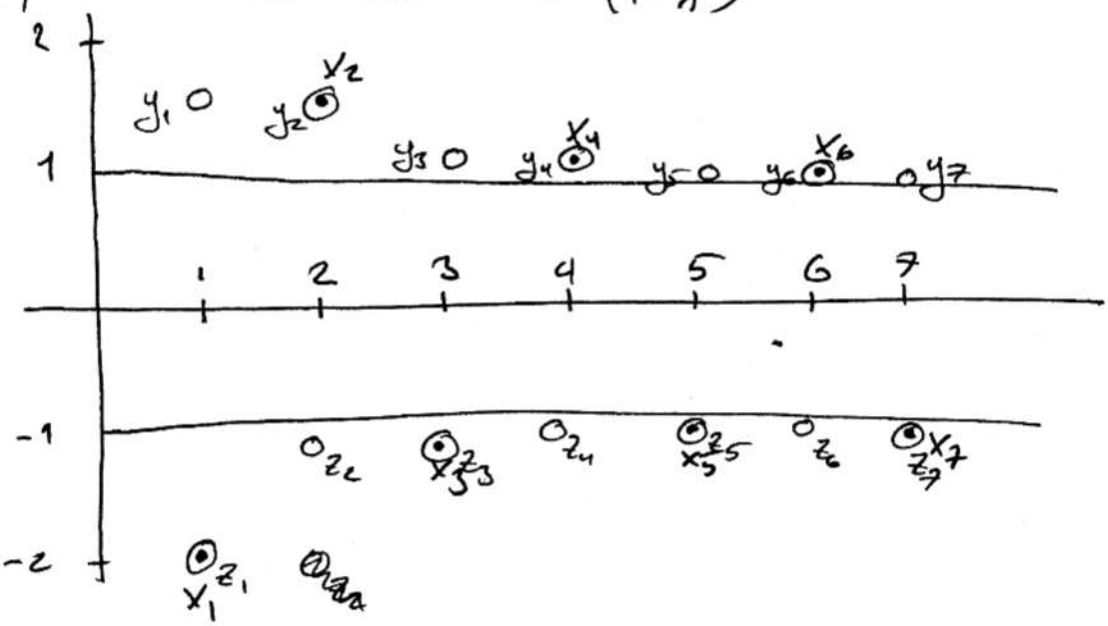
Not every sequence has a limit.
However, every sequence has what we call limsup & liminf.

Given a sequence $(x_n)_{n=1}^{\infty}$, set $y_n = \sup\{x_k : k \geq n\}$
$z_n = \inf\{x_k : k \geq n\}$

Then y_n is monotone decreasing $\Rightarrow$ $\lim y_n$ exists set $\lim y_n = \limsup$.

z_n is monotone increasing $\Rightarrow$ $\lim z_n$ exists. Set $\limsup x_n = \lim z_n$

Example: $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$



$\limsup_{n\to\infty} x_n = \lim y_n = 1$ $\qquad$ $\liminf x_n = \lim z_n = -1$

Equivalent definitions:

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$$
$$= \inf\{\alpha : \text{Only finitely many } x_n \text{ are larger than } \alpha\}$$
$$= \sup\{\alpha : \text{there exists a subseq } (n_j)_{j=1}^{\infty} \text{ such that } \alpha = \lim_{j\to\infty} x_{n_j}\}$$

Lemma For *any* sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$:

* $\limsup x_n$ and $\liminf x$ exist (but *may* equal $\pm\infty$)
* $\liminf x_n \leq \limsup x_n$
* (x_n) is convergent $\Leftrightarrow$ $\limsup x_n = \liminf x_n$

  In this case, $\lim x_n = \limsup x_n = \liminf x_n$, of course.

CONTINUITY

Def^n Let X and Y be metric spaces, and suppose $f: X \to Y$.

* We say that f is CONTINUOUS at x if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$.
* We say that f is CONTINUOUS if it is cont. at every $x \in X$.
* We say that f is SEQUENTIALLY CONTINUOUS AT x if for every seq. (x_n) s.t. $x_n \to x$, we have $f(x_n) \to f(x)$
* We say that f is 'sequentially continuous' if f is seq cont. at every $x \in X$.

Note: We will soon prove that seq. cont $\Leftrightarrow$ cont.

Example Let X and Y be metric spaces.

Suppose that X has the discrete metric, $d_X(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$

Then *any* function $f: X \to Y$ is seq. cont

Proof: Suppose $x_n \to x$ in X. We need to prove that $f(x_n) \to f(x)$ in Y.

But if $x_n \to x$ in X, $\exists N$ s.t. $n \geq N \Rightarrow x_n = x$.

Then for $n \geq N$, $f(x_n) = f(x)$ so $f(x_n) \to f(x)$.

OPEN & CLOSED SETS

Def^n Let $(\mathbb{X}, d)$ be a metric space.

* For $c \in \mathbb{X}$, $r \in [0,\infty)$, define $B_r(c) = \{x \in \mathbb{X} : d(x,c) < r\}$ ← "open ball"

~~* for $c \in \mathbb{X}$, $r \in [0,\infty)$, define $\overline{B_r(c)} = \{x \in \mathbb{X} : d(x,c) \leq$~~

* A set $G \subseteq \mathbb{X}$ is OPEN if $\forall x \in G$ $\exists r > 0$ s.t. $B_r(x) \subseteq G$
* A set $F \subseteq \mathbb{X}$ is CLOSED if $F^c = \mathbb{X} \setminus F$ is open.
* The BOUNDARY of a set $\Omega \subseteq \mathbb{X}$ is the set of all points $x \in \mathbb{X}$ such that for any $\varepsilon > 0$, $B_\varepsilon(x)$ contains points in both Ω and Ω^c

Prop^n Let $(\mathbb{X}, d)$ be a metric space.

(i) $\mathbb{X}$ and $\emptyset$ are both open and closed.
(ii) A finite intersection of open sets is open.
(iii) Any union of open sets is open.
(iv) A finite union of closed sets is closed.
(v) Any intersection of closed sets is closed.

Note that the requirements of finiteness above is necessary.

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1\right) = [0,1)$$

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1+\frac{1}{n}\right) = [0,1]$$

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1-\frac{1}{n}\right] = (0,1)$$

Proof of (iii) Let $\{G_\alpha\}_{\alpha\in A}$ be a collection of open sets and set $G = \bigcup_{\alpha\in A} G_\alpha$.
Fix an $x \in G$. $\exists\, \beta \in A$ s.t. $x \in G_\beta$.
Since G_β is open, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq G_\beta$.
Since $G_\beta \subseteq G$, it follows that $B_\varepsilon(x) \subseteq G$.

Proof of (v): Let $\{F_\alpha\}_{\alpha\in A}$ be a collection of closed sets and set $F = \bigcap_{\alpha\in A} F_\alpha$.
That F is closed follows immediately from (iii) since
$F^c = \bigcup_{\alpha\in A} F_\alpha^c$ and all F_α^c are open.

Prop^n ~~Def^n~~

Let X & Y be metric spaces, and let f map X to Y.
TFAE: (a)* f is ε-δ-cont.
(b)* f is seq. cont.
(c) f is open-set cont.

Proof: We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a)

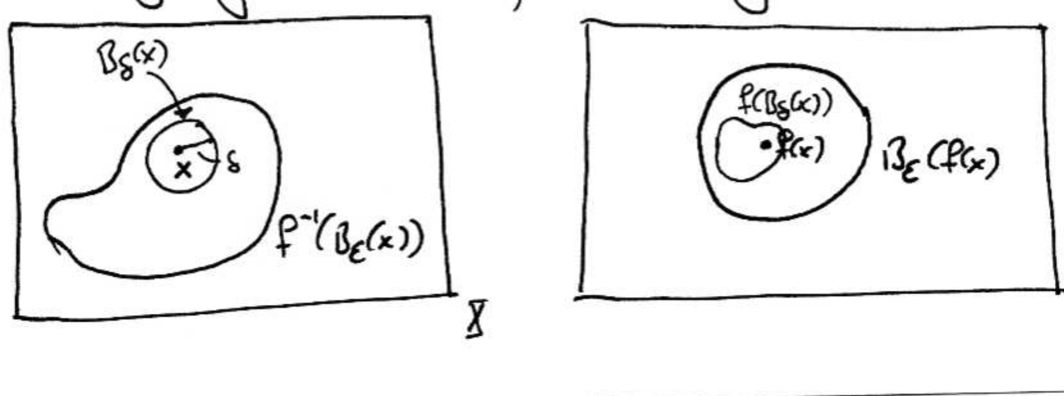
(a)$\Rightarrow$(b) Suppose that f is ε-δ-cont.
We need to prove that if $x_n \to x$, then $f(x_n) \to f(x)$.
Suppose that $x_n \to x$. Fix $\varepsilon > 0$.
f is ε-δ cont at $x \Rightarrow \exists\, \delta$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.
Since $x_n \to x$, $\exists\, N$ s.t. $n \geq N \Rightarrow x_n \in B_\delta(x)$.
Then for $n \geq N$, $f(x_n) \in B_\varepsilon(f(x))$.

Note: The prop^n says that in a metric space, the three notions of continuity are equivalent. Henceforth, we will simply say "continuous".

<u>(b)⇒(c)</u> Suppose that f is NOT open-set cont.
We will construct a seq (x_n) s.t. $x_n \to x$ for some x,
but $f(x_n) \not\to f(x)$.
Since f is not open-set cont, $\exists$ an open set $G \subseteq Y$ s.t. $H = f^{-1}(G)$ is not open.
Since H is not open, $\exists x \in H$ s.t. $B_\varepsilon(x) \cap H^c$ is non-empty for every $\varepsilon > 0$.
For $n = 1, 2, 3, \dots$ pick $x_n \in B_{1/n}(x) \cap H^c$.
Then $x_n \to x$.
However, since $f(x) \in G$, and G is open,
$\exists \varepsilon$ s.t. $B_\varepsilon(f(x)) \in G$.
Since $f(x_n) \notin G$, it follows that $d(f(x_n), f(x)) > \varepsilon \ \forall n$
and so $f(x_n)$ cannot converge to $f(x)$.

<u>(c)⇒(a)</u> Suppose that f is open-set cont.
We will prove that f is ε-δ-cont. Fix an $x \in X$.
Pick any $\varepsilon > 0$.
Since $B_\varepsilon(f(x))$ is open in Y, $f^{-1}(B_\varepsilon(f(x)))$ is open in X.
Since $x \in f^{-1}(B_\varepsilon(f(x)))$, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$
But then $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

<u>Note</u>: Drawing figures for the proof is a good idea. For instance, (c)⇒(a)



Def^n Suppose that Ω is a subset of a metric space X.
The closure of Ω is the set $\overline{\Omega} = \{x \in X : \exists (x_n) \subset \Omega \text{ s.t. } x_n \to x\}$.

Claim Letting $\partial\Omega$ denote the boundary of Ω, we have $\overline{\Omega} = \Omega \cup \partial\Omega$.

Claim $\overline{\Omega}$ = the intersection of all closed sets containing Ω.

Claim Ω is closed $\Leftrightarrow$ $\Omega = \overline{\Omega}$.

Example Let X denote the metric space $\mathbb{R}$ with the usual metric.
Let $\Omega = \mathbb{Q}$. Then $\overline{\Omega} = \mathbb{R} = X$.

Example Let X denote the set $\mathbb{R}$ and let $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y. \end{cases}$
Let Ω be any subset of X. Then $\overline{\Omega} = \Omega$.
(Note that in the discrete metric, ANY set is closed.)

Def^n Let X be a metric space.
A subset Ω is said to be dense in X if $\overline{\Omega} = X$.

~~Example Under th~~

Def^n If a metric space has a dense countable subset, then we say that the metric space is seperable.

Example $\mathbb{R}$ equipped with the usual metric is seperable since $\mathbb{Q}$ is dense.

Example $\mathbb{R}$ equipped with the discrete metric is NOT seperable since no ~~sep~~ subset is dense (except $\mathbb{R}$ itself).

Example Let X denote the set of all sequences $x=(x_1, x_2, x_3, \dots)$ such that (i) $x_n \in \mathbb{Q}$ $\forall n$, and (ii) only finitely many x_n's are non-zero. Define for $x=(x_1, x_2, \dots)$ & $y=(y_1, y_2, y_3, \dots)$ the metric $d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2 \right)^{1/2}$.

~~Then X is~~

Note that X is countable (to see this, note that $X = \bigcup_{n=1}^{\infty} X_n$, where X_n is the set of all sequences $(x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$ where $x_j \in \mathbb{Q}$ $\forall j$, then each X_n is countable since it can be identified with $\mathbb{Q}^n$, and thus X is a countable union of countable sets).

Next ~~set~~ let $\tilde{X}$ denote the set of all sequences $x=(x_1, x_2, \dots)$ such that $x_n \in \mathbb{R}$ and $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. Equip $\tilde{X}$ with the same metric as X.

~~Note that~~ X is dense in $\tilde{X}$. To prove this, fix any $x \in \tilde{X}$. $x=(x_1, x_2, \dots)$

Fix a j. (set $\varepsilon = \frac{1}{j}$) Pick N_j s.t. $\sum_{n=N_j+1}^{\infty} |x_n|^2 < \frac{\varepsilon^2}{2}$.

Next, pick for $n=1,2,\dots,N_j$ numbers $x_n^{(j)} \in \mathbb{Q}$ such that $|x_n - x_n^{(j)}| < \frac{\varepsilon}{\sqrt{2N_j}}$, and set $x^{(j)} = (x_1^{(j)}, \dots, x_{N_j}^{(j)}, 0, 0, \dots)$.

Then $x^{(j)} \in X$, and $d(x, x^{(j)}) = \Big(\underbrace{\sum_{n=1}^{N_j} |x_n - x_n^{(j)}|^2}_{< \varepsilon^2/2} + \underbrace{\sum_{n=N_j+1}^{\infty} |x_n|^2}_{< \varepsilon^2/2} \Big)^{1/2} < \varepsilon = \frac{1}{j}$

Thus $x^{(j)} \to x$. ~~and so x~~

COMPLETION OF A METRIC SPACE

It sometimes happen that we define a set X and a metric d and find that the resulting space (X, d) is not complete. This is highly inconvenient.

It turns out to be possible to "add the missing elements" and obtain a new space $(\tilde{X}, \tilde{d})$ that is complete. This space is called the COMPLETION of (X, d).

- It is in a certain sense unique.

Example: The set of real numbers $\mathbb{R}$ can be defined by first defining the rational numbers $\mathbb{Q}$ and then form the completion of $\mathbb{Q}$ w.r.t. the metric $d(x,y) = |x-y|$.

Example: The spaces X & $\tilde{X}$ on page 13.

Example: The homework problem where $I = [0,1]$ and X = the space of all continuous functions on I, and $d(f,g) = \left(\int_I |f(x) - g(x)|^2 dx \right)^{1/2}$.

Then $\tilde{X} = L^2(I)$ = the space of all Lebesgue-measurable functions f s.t. $\int_0^1 |f(x)|^2 dx < \infty$.

Caveat: In $L^2(I)$, two functions f and g are considered identical if $\int |f(x) - g(x)|^2 dx = 0$. To be precise, an element in $L^2(I)$ is an equivalence class of functions.