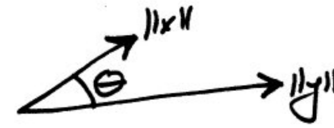
Hilbert Spaces

A Hilbert space is a Banach space with an "inner product".
In other words, a Banach space where we can define a concept of angles between vectors, recall in $\mathbb{R}^n$:

$$\cancel{\cos =} \qquad (x,y) = \|x\| \, \|y\| \cos\theta$$



In this section, all linear spaces will be COMPLEX.

Def^n Let $\mathfrak{X}$ be a complex linear space.
We say that a map $(\cdot,\cdot) : \mathfrak{X}\times\mathfrak{X} \to \mathbb{C}$ is an INNER PRODUCT if

(1) $(x, \alpha y + \beta z) = \alpha(x,y) + \beta(x,z) \qquad \forall \alpha,\beta\in\mathbb{C} \quad x,y,z\in\mathfrak{X}$

(2) $(x,y) = \overline{(y,x)} \qquad \forall x,y\in\mathfrak{X} \qquad (\Rightarrow (x,x)\in\mathbb{R})$

(3) $(x,x) \geq 0 \qquad \forall x \in \mathfrak{X}$

(4) $(x,x) = 0 \iff x = 0.$

If such a map exists, we say that $\mathfrak{X}$ is an inner product space.

Lemma $|(x,y)| \leq \sqrt{(x,x)(y,y)} \quad \forall x,y\in\mathfrak{X}$ $\left(\begin{array}{l}\text{Setting } \|x\| = \sqrt{(x,x)}, \text{ we have} \\ |(x,y)| \leq \|x\| \, \|y\|\end{array}\right)$

Proof: If $(x,y)=0$, then obvious, otherwise ~~Let~~ for any $\beta\in\mathbb{C}$, we have

$$0 \leq (x-\beta y, x-\beta y) = (x,x) - (x,\beta y) - (\beta y, x) + (\beta y, \beta y) =$$

$$= (x,x) - 2\operatorname{Re}\beta(x,y) + |\beta|^2 (y,y) = \qquad \text{set } \beta = \frac{\overline{(x,y)}}{|(x,y)|} t \quad \text{for } t\in\mathbb{R}$$

$$= (x,x) - 2t\,|(x,y)| + t^2 (y,y) = (y,y)\left[t^2 - 2t\frac{|(x,y)|}{(y,y)} + \frac{(x,x)}{(y,y)} \right] =$$

$$= (y,y)\left[\left(t - \frac{|(x,y)|}{(y,y)}\right)^2 - \frac{|(x,y)|^2}{(y,y)^2} + \frac{(x,x)}{(y,y)} \right]$$

$$\Rightarrow \frac{(x,x)}{(y,y)} - \frac{|(x,y)|^2}{(y,y)^2} \geq 0 \quad \Rightarrow \quad |(x,y)|^2 \leq (x,x)(y,y)$$

Lemma Any inner product space is a normed linear space, with the norm $\|x\| = \sqrt{(x,x)}$

Proof * $\|\alpha x\| = \sqrt{(\alpha x, \alpha x)} = \sqrt{\alpha\bar{\alpha}(x,x)} = |\alpha|\sqrt{(x,x)} = |\alpha|\,\|x\|$

* $\|x\| = 0 \iff (x,x) = 0 \iff x = 0$

* $\|x+y\|^2 = (x+y, x+y) = \|x\|^2 + (x,y) + (y,x) + \|y\|^2 \leq$

$$\leq \|x\|^2 + \|x\|\,\|y\| + \|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

Def^n A complete inner product space is called a Hilbert space.

Lemma If $x_n \to x$ & $y_n \to y$, then $(x_n, y_n) \to (x,y)$

Proof $|(x_n, y_n) - (x,y)| = |(x_n, y_n - y) + (x_n - x, y)| \leq$

$$\leq \|x_n\|\,\|y_n - y\| + \|x_n - x\|\,\|y\| \to 0 \quad \text{as } n \to \infty$$

(Recall that if $x_n \to x$, then $\sup_{n\to\infty} \|x_n\| < \infty$)

~~Rem~~ **Corollary** The inner product is a continuous map $\mathbb{X} \times \mathbb{X} \to \mathbb{C}$.

Lemma If $\mathbb{X}$ is an inner product space, then

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x,y \in \mathbb{X} \qquad \text{(PAR)}$$

Proof $\|x+y\|^2 + \|x-y\|^2 = (x+y, x+y) + (x-y, x-y) =$

$$= \|x\|^2 + (x,y) + (y,x) + \|y\|^2 + \|x\|^2 - (x,y) - (y,x) + \|y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

<u>Lemma</u> If $\mathfrak{X}$ is a NLS where (PAR) holds, then
$$(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right)$$
defines an inner product on $\mathfrak{X}$ such that $\|x\| = \sqrt{(x,x)}$.

<u>Proof:</u> Homework. ~~#~~

<u>Remark:</u> A Banach space can be equipped with an inner product (and thus be turned into a Hilbert space) if and only if its norm satisfies (PAR).

<u>Remark:</u> An inner product is uniquely defined by the norm it induces. In particular, it is uniquely defined by its diagonal values.

Examples of Hilbert spaces:

(1) $\mathbb{C}^n$

(2) l^2

(3) Set $I = [0,1]$ $\mathfrak{X}$ = the set of continuous funcs on I.
Set $(f,g)_2 = \int_0^1 \overline{f(x)}\, g(x)\,dx \Rightarrow \|f\| = \left(\int_0^1 |f(x)|^2 dx\right)^{1/2}$
$\mathfrak{X}$ is an inner product space but <u>not</u> a Hilbert space.
The completion of $\mathfrak{X}$ is the Hilbert space $L^2(I)$.
It consists of all functions such that $|f(x)|^2$ is Lebesgue integrable.

(4) Set $I = [0,1]$, $\mathfrak{X}$ = the set of infinitely differentiable functions on I.
Set $(f,g)_{H^n} = \sum_{j=0}^{n} \int_0^1 \overline{f^{(j)}(x)}\, g^{(j)}(x)\,dx \Rightarrow \|f\|_{H^n} = \left(\sum_{j=0}^{n} \int_0^1 |f^{(j)}(x)|^2 dx\right)^{1/2}$
$\mathfrak{X}$ is an inner product space.
The completion of $\mathfrak{X}$ is called H^n, it is a "Sobolev" space.

Orthogonality

Def^n Let $\mathfrak{X}$ be an inner product space.
If $x, y \in \mathfrak{X}$ and $(x,y)=0$, we say that x & y are ORTHOGONAL, $x \perp y$.
Let A be a subset of $\mathfrak{X}$. The ORTHOGONAL COMPLEMENT of A is
$$A^\perp = \{ y \in \mathfrak{X} : (x,y)=0 \quad \forall x \in A \}$$

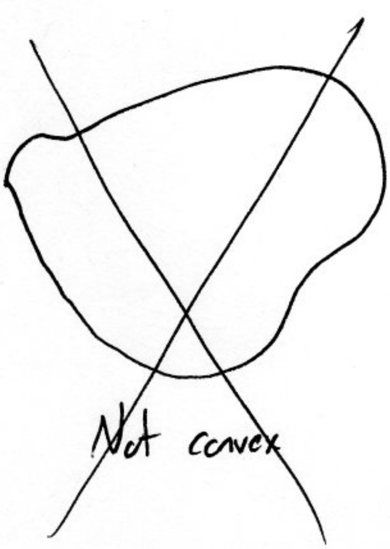
Lemma Suppose that $(x_j)_{j=1}^n \subset \mathfrak{X}$ & $x_j \perp x_k$ for $j \neq k$.
Then $\| \sum_{j=1}^n x_j \|^2 = \sum_{j=1}^n \|x_j\|^2$

Proof: ~~Trivial~~ $\| \sum_{j=1}^n x_j \|^2 = \left(\sum_{j=1}^n x_j, \sum_{k=1}^n x_k \right) = \sum_{j=1}^n \sum_{k=1}^n (x_j, x_k) = \sum_{j=1}^n \|x_j\|^2$

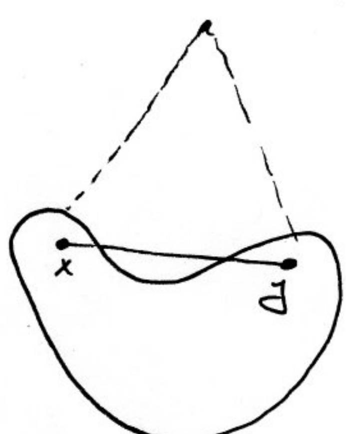
Lemma: Let A be a ~~subspace~~ subset of a Hilbert space $\mathfrak{X}$.
Then $A^\perp$ is a <u>closed</u> subspace of $\mathfrak{X}$.

Proof: Homework.

Def^n Let $\mathfrak{X}$ be a linear space, and M a subset of $\mathfrak{X}$.
We say that M is <u>convex</u> if $\forall x, y \in M$, $\alpha x + (1-\alpha) y \in M$ for $0 \le \alpha \le 1$



Not convex



Not convex

Convex

Lemma Let M be a closed convex set in a Hilbert space $\mathfrak{X}$, and let x be a point in M^c.

There exists a unique element $\hat{y} \in M$ s.t. $\|x - \hat{y}\| = \inf_{y \in M} \|x - y\|$

Proof Set $d = \inf_{y \in M} \|x - y\|$. Pick $(y_n) \in M$ s.t. $\|y_n - x\| \to d$.

We will prove that (y_n) is a Cauchy seq.:

Fix $\varepsilon > 0$. Pick N s.t. $n \geq N \Rightarrow \|x - y_n\| \leq d + \varepsilon$. For $n \geq N$:

$$(\text{PAR}) \Rightarrow \|y_m - y_n\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - \|2x - y_n - y_m\|^2 \qquad (*)$$

Now $\|2x - y_n - y_m\| = 2\|x - \frac{y_n + y_m}{2}\| \geq 2d$ since $\frac{y_n + y_m}{2} \in M$.

So $(*) \Rightarrow \|y_m - y_n\|^2 \leq 2(d+\varepsilon)^2 + 2(d+\varepsilon)^2 - 4d^2 = 8d\varepsilon + 4\varepsilon^2 \to 0$ as $\varepsilon \to 0$.

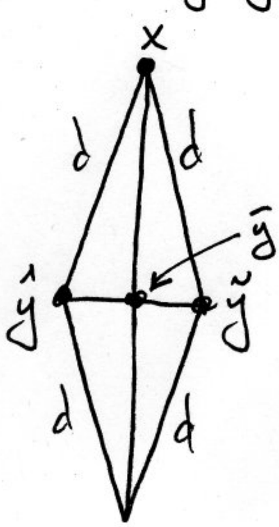
Since $\mathfrak{X}$ is complete & M is closed, $\exists\ \hat{y} \in M$ s.t. $y_n \to \hat{y}$.

$\|x - \hat{y}\| = \lim_{n \to \infty} \|x - y_n\| = d$.

It only remains to prove uniqueness.

Suppose $\hat{y}$ & $\tilde{y}$ are s.t. $d = \|x - \hat{y}\| = \|x - \tilde{y}\|$

Set $a = \|\hat{y} - \tilde{y}\|$, $\bar{y} = \frac{1}{2}(\hat{y} + \tilde{y})$, and $b = \|x - \bar{y}\| \geq d$



Parallelogram law $\Rightarrow a^2 + 4b^2 = 4d^2$

But since $b \geq d$, this implies that $a = 0$

so $\hat{y} = \tilde{y}$

Def^n Let $\mathfrak{X}$ be a vector space, and let A and B be subspaces of $\mathfrak{X}$. We say that $\mathfrak{X} = A \oplus B$ (a "direct" sum) if for any $x \in \mathfrak{X}$ there exist <u>unique</u> $y \in A$, $z \in B$ such that $x = y + z$.

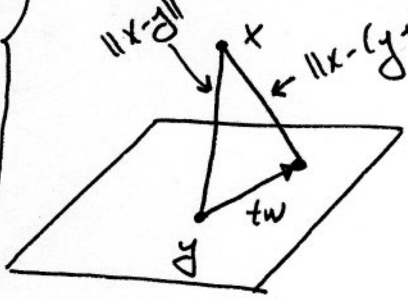
Thm Let $\mathcal{M}$ be a closed subspace of a Hilbert Space $\mathfrak{X}$. Then ~~M~~ $\mathfrak{X} = \mathcal{M} \oplus \mathcal{M}^\perp$.

Proof Let $\mathcal{M}$ be as specified and pick $x \in \mathcal{M}^c$.

existence

Let $y \in \mathcal{M}$ be the unique element s.t. $\|x-y\| = \inf_{y' \in \mathcal{M}} \|x-y'\|$.

Set $z = x - y$. We need to prove that $z \in \mathcal{M}^\perp$. Pick any $w \in \mathcal{M}$:



Set $f(t) = \|x - (y+tw)\|^2 = \|z\|^2 - 2t\,\mathrm{Re}(z,w) + t^2\|w\|^2$

$f(t)$ has a minimum at $t=0$. Thus

$0 = f'(0) = -2\,\mathrm{Re}(z,w) \Rightarrow \mathrm{Re}(z,w) = 0$

To prove that $\mathrm{Im}(z,w) = 0$, consider $g(t) = \|x - (y + itw)\|^2$

uniqueness

It remains only to prove uniqueness.

Suppose that $y + z = y' + z'$ where $y, y' \in \mathcal{M}$, $z, z' \in \mathcal{M}^\perp$.

Then $y - y' = z' - z \Rightarrow \|y-y'\|^2 = (\underbrace{y-y'}_{\in \mathcal{M}}, \underbrace{z'-z}_{\in \mathcal{M}^\perp}) = 0 \Rightarrow y = y' \Rightarrow z = z'$

$\blacksquare$

In other words, given any closed subspace $\mathcal{M}$, any vector can uniquely be decomposed as $x = y + z$ where $y \in \mathcal{M}$, $z \in \mathcal{M}^\perp$.

Moreover:

$$\|x-y\| = \inf_{y' \in \mathcal{M}} \|x - y'\|$$

$$\|x-z\| = \inf_{z' \in \mathcal{M}^\perp} \|x - z'\|$$

Def^n A projection on a linear space is a ~~lin~~ linear map P s.t. $P^2=P$.
On a Banach space, a projection must also be continuous.

Given any ~~linear~~ closed linear subspace ~~X~~ M, set $Px=y$ & $Qx=z$ where $x=y+z$ and $y\in M$, $z\in M^\perp$.

P and Q are both projections, $\|P\|=\|Q\|=1$, $PQ=QP=0$ & $P+Q=I$
$\mathrm{Ran}(P)=M$ & $\mathrm{Nul}(P)=M^\perp$.

Conversely, we have:

Thm Let P be a proj^n on $\underbrace{H}_{\text{a Hilbert Space}}$ with range M and nullspace N. Then

$$M\perp N \iff (Px,y)=(x,Py)\ \forall x,y\in X$$

On a Banach Space, we have the following theorems:

Thm Let P be a proj^n on a Banach space X, and set $M=\mathrm{Ran}(P)$, $N=\mathrm{Nul}(P)$
Then M and N are closed linear ~~sup~~ subspaces such that $X=M\oplus N$.

Thm Let X be a Banach space, and let M and N be closed linear subspaces such that $X=M\oplus N$. For $x\in X$, set $Px=y$ where $x=y+z$, $y\in M$, $z\in N$. Then P is a proj^n such that $M=\mathrm{Ran}(P)$, $N=\mathrm{Nul}(P)$.

Remark: The principal difference between Banach and Hilbert spaces in this regard is that for Hilbert spaces, we only need one subspace, namely M. Then $N=M^\perp$ always exists.
Not so for Banach spaces. There exist examples of Banach spaces X and closed linear subspaces M s.t. M is not the range of any (cont.) proj^n.