Applied Analysis (APPM 5440): Final exam

1:30pm – 4:00pm, Dec. 14, 2009. Closed books.

Problem 1: (20p) Set $I = [0, 1]$. Prove that there is a continuous function u on I such that

$$u(x) - \frac{1}{5}\int_0^x \sin\big(u(t)^2\big)\,dt = \cos(x), \qquad x \in I. \tag{1}$$

Solution: Define the operator T via

$$[Tu](x) = \frac{1}{5}\int_0^x \sin\big(u(t)^2\big)\,dt + \cos(x).$$

Clearly Tu is continuous whenever u is continuous so T maps $C(I)$ to $C(I)$. Now (1) takes the form

$$u = Tu. \tag{2}$$

Define $\Omega = \{u \in C(I)\,:\, ||u|| \leq M\}$ where the norm is the uniform norm and M is a real number to be determined. Ω is a closed subset of the complete space $C(I)$, so if we can determine M so that T maps Ω to Ω, and T is a contraction on Ω, then the contraction mapping theorem will apply.

First we observe that $|\sin\alpha| \leq |\alpha|$ and $|\sin\alpha - \sin\beta| \leq |\alpha - \beta|$ for any real numbers α and β.

Ensure that T maps Ω to Ω: Suppose that $u \in \Omega$. Then

$$||Tu|| = \sup_{x\in I}\left|\frac{1}{5}\int_0^x \sin(u(t)^2)\,dt + \cos(x)\right| \leq \sup_{x\in I}\left(\frac{1}{5}\int_0^x |\sin(u(t)^2)|\,dt + |\cos(x)|\right) \leq \frac{1}{5}\int_0^1 1\,dt + 1 = \frac{6}{5}$$

We see that $Tu \in \Omega$ if $M \geq 6/5$.

Ensure that T is a contraction: If $u, v \in \Omega$, then

$$\begin{aligned}
||Tu - Tv|| &= \sup_{x\in I}\left|\frac{1}{5}\int_0^x \big(\sin u(t)^2 - \sin v(t)^2\big)\right|\,dt \leq \frac{1}{5}\int_0^1 |\sin u(t)^2 - \sin v(t)^2|\,dt \\
&\leq \frac{1}{5}\int_0^1 |u(t)^2 - v(t)^2|\,dt = \frac{1}{5}\int_0^1 |u(t) - v(t)|\,|u(t) + v(t)|\,dt \\
&\leq \frac{1}{5}\int_0^1 |u(t) - v(t)|\,(|u(t)| + |v(t)|)\,dt \leq \frac{1}{5}\sup_{x\in I}\int_0^1 ||u - v||\,(M + M)dt = \frac{2M}{5}||u - v||.
\end{aligned}$$

We see that T is a contraction if $M < 5/2$.

Conclusion: Picking, say, $M = 2$, we see that T becomes a contraction on the complete metric space Ω. Consequently, (2) has a unique solution.

Problem 2: (25p) Let X be a set.

(a) (6p) State the definitions of a metric on X and a topology on X.

(b) (4p) Given a metric d on X, define the topology $\mathcal{T}$ induced by d.

(c) (8p) Prove that the $\mathcal{T}$ that you defined in (b) satisfies the axioms of a topology.

(d) (7p) Set $Y = \mathbb{R}^2$, and define on $Y \times Y$ the function

$$b(x, y) = b([x_1, x_2], [y_1, y_2]) = \left(|x_1 - y_1|^{1/2} + |x_2 - y_2|^{1/2}\right)^2.$$

Is (Y, b) a metric space? Motivate.

Solution:

(a) See text book.

(b) A subset G of X belongs to $\mathcal{T}$ if and only if the following condition holds:
For any $x \in G$, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq G$, where $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$.

(c) $\emptyset$ and X obviously satisfy the criterion in (b).

Let $\{G_\alpha\}_{\alpha \in A}$ be an arbitrary collection of sets in $\mathcal{T}$. Set $G = \bigcup_{\alpha \in A} G_\alpha$. We need to show that $G \in \mathcal{T}$. Suppose $x \in G$. Then $x \in G_\beta$ for some $\beta \in A$. Since $G_\beta \in \mathcal{T}$, there is a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq G_\beta$. But then $B_\varepsilon(x) \subseteq G_\beta \subseteq \bigcup_{\alpha \in A} G_\alpha = G$.

Let $\{G_j\}_{j=1}^n$ be a finite collection of sets in $\mathcal{T}$. Set $G = \bigcap_{j=1}^n G_j$. We need to show that $G \in \mathcal{T}$. Pick $x \in G$. Then $x \in G_j$ for every j, so for each j we can pick an $\varepsilon_j > 0$ such that $B_{\varepsilon_j}(x) \subseteq G_j$. Set $\varepsilon = \min_{1 \leq j \leq n} \varepsilon_j$. Since the min is over a finite number of elements, $\varepsilon > 0$. Moreover, $B_\varepsilon(x) \subseteq G_j$ for every j, so $B_\varepsilon(x) \subseteq G$.

(d) No, the given b does not satisfy the triangle inequality. Consider the points

$$x = [1, 0], \qquad y = [0, 1], \qquad z = [1, 1].$$

We have

$$b(x, y) = \left(1^{1/2} + 1^{1/2}\right)^2 = 4,$$

and

$$b(x, z) + b(z, y) = \left(0^{1/2} + 1^{1/2}\right)^2 + \left(0^{1/2} + 1^{1/2}\right)^2 = 2.$$

Problem 3: (10p)

(a) (5p) State the Hahn-Banach theorem.

(b) (5p) Define what it means for a sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X to converge weakly.

Solution:

(a) Let M be a linear subspace of a NLS X, and let ψ be a bounded linear functional on M. Then there exists a bounded linear functional φ on X such that

$$\varphi(x) = \psi(x), \qquad \forall\, x \in M,$$

and

$$||\varphi||_{X^*} = ||\psi||_{M^*}.$$

Alternative formulation: Every bounded linear functional of a subspace of a NLS can be extended to the entire space without enlarging the norm.

(b) A sequence $(x_n)_{n=1}^{\infty}$ converges weakly if there exists an element $x \in X$ such that

$$\varphi(x_n) \to \varphi(x) \qquad \text{for every } \varphi \in X^*.$$

Problem 4: (25p) Let H be a Hilbert space with a closed convex subset M.

(a) (13p) Suppose that $x \in H$ and that $x \notin M$. Prove that there exists a unique $z \in M$ such that

$$||x - z|| = \inf_{y \in M} ||x - y||.$$

(b) (12p) Now consider the particular case of $H = L^2(I)$ where $I = [0, 1]$. The set H is equipped with the usual inner product $(f, g) = \int_0^1 \overline{f(t)}\, g(t)\, dt$. Let M denote the linear space of polynomials of degree two or less, and set $f(t) = t^3$. Evaluate

$$d = \text{dist}(M, f) = \inf_{g \in M} ||f - g||.$$

Solution:

(a) See lecture notes for Chapter 6. (Observe that M is only a sub<u>set</u> of H, not a sub<u>space</u>.)

(b) The unique minimizer z assured in part (a) takes the form

$$z(t) = a + b\,t + c\,t^2$$

where a, b, and c are some numbers to determined. We know that $f - z \in M^{\perp}$. Since the functions $u_1(t) = 1$, $u_2(t) = t$, and $u_3(t) = t^2$ form a basis for M, the condition that $f - z \in M^{\perp}$ is satisfied iff

$$\begin{aligned}
0 = (f - z, u_1) &= \int_0^1 \left(t^3 - a - b\,t - c\,t^2\right) dt = \frac{1}{4} - a - \frac{1}{2}\,b - \frac{1}{3}\,c, \\
0 = (f - z, u_2) &= \int_0^1 \left(t^4 - a - b\,t^2 - c\,t^3\right) dt = \frac{1}{5} - \frac{1}{2}\,a - \frac{1}{3}\,b - \frac{1}{4}\,c, \\
0 = (f - z, u_3) &= \int_0^1 \left(t^5 - a - b\,t^3 - c\,t^4\right) dt = \frac{1}{6} - \frac{1}{3}\,a - \frac{1}{4}\,b - \frac{1}{5}\,c.
\end{aligned}$$

Solving the linear system

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/5 \\ 1/6 \end{bmatrix}$$

we find that

$$a = \frac{1}{20}, \qquad b = -\frac{3}{5}, \qquad c = \frac{3}{2}.$$

Now the laborious part is to evaluate the norm of the residual. After some work, we find

$$\text{dist}(M, f) = ||z|| = \left(\int_0^1 \left(t^3 - \frac{1}{20} + \frac{3}{5}t - \frac{3}{2}t^2\right)^2 dt\right)^{1/2} = \cdots = \frac{1}{\sqrt{2800}} = \frac{1}{20\sqrt{7}}.$$

Alternate solution: It is possible to use Gram-Schmidt to orthonormalize $\{u_1,\, u_2,\, u_3\}$ to form an orthonormal set $\{v_1,\, v_2,\, v_3\}$; this is a bit of work, but results in the functions

$$v_1(t) = 1, \qquad v_2(t) = 2\sqrt{3}(t - 1/2), \qquad v_3(t) = 6\sqrt{5}(t^2 - t + 1/6).$$

Then $z = (v_1, f)\, v_1 + (v_2, f)\, v_2 + (v_3\, f)\, v_3$, and since $f - z \in M^{\perp}$, Pythagoras theorem yields

$$||f - z||^2 = ||f||^2 - ||z||^2 = \frac{1}{7} - |(v_1, f)|^2 - |(v_2, f)|^2 - |(v_2, f)|^2$$

$$= \frac{1}{7} - \frac{1}{16} - \left(\frac{3\sqrt{12}}{40}\right)^2 - \left(\frac{\sqrt{5}}{20}\right)^2 = \frac{1}{2800}.$$

Problem 5: (20p) Let (X, d) be a compact metric space. Let $C_{\mathrm{b}}(X)$ denote the set of all bounded real-valued continuous functions on X, equipped with the uniform norm,

$$||f||_{\mathrm{u}} = \sup_{x \in X} |f(x)|.$$

Prove that $C_{\mathrm{b}}(X)$ is complete.

Solution: The assumption that X is compact is a red herring — this property is not required for the statement to be true.

Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in (X, d). We will construct a limit function, and then prove that it is bounded, that it is indeed the limit of the sequence in the uniform norm, and finally that it is continuous.

Step 1 — construct the limit point f: Fix $x \in X$. Since $|f_n(x) - f_m(x)| \leq ||f_n - f_m||$ and $(f_n)_{n=1}^{\infty}$ is Cauchy, the sequence $(f_n(x))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, the sequence is convergent, we can therefore define a function f via

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Step 2 — prove that f is bounded: We have

$$\sup_{x \in X} |f(x)| = \sup_{x \in X} \left(\lim_{n \to \infty} |f_n(x)|\right) \leq \liminf_{n \to \infty} \left(\sup_{x \in X} |f_n(x)|\right) = \liminf_{n} ||f_n|| < \infty,$$

where in the last step we used that (f_n) is Cauchy, and therefore bounded.

Step 3 — prove that $f_n \to f$ uniformly: Fix $\varepsilon > 0$. Pick N such that $||f_m - f_n|| < \varepsilon/2$ when $m, n \geq N$. Then for $n \geq N$, we have

$$||f_n - f|| = \sup_{x \in X} |f_n(x) - f(x)| = \sup_{x \in X} \left(\lim_{m \to \infty} |f_n(x) - f_m(x)|\right)$$

$$\leq \liminf_{m \to \infty} \left(\sup_{x \in X} |f_n(x) - f_m(x)|\right) = \liminf_{m \to \infty} ||f_n - f_m|| \leq \varepsilon/2 < \varepsilon.$$

Step 4 — prove that f is continuous: This follows directly from the fact that each f_n is continuous and $f_n \to f$ uniformly (since uniform convergence preserves continuity).

Steps 2 and 4 prove that $f \in C_{\mathrm{b}}(X)$, and step 3 proves that f is the limit point of (f_n). The proof is therefore complete.