Homework set 10 — APPM5440

Problem 5.4:

You can compute the eigenvalues of A using standard techniques. This directly leads to a formula for $r(A)$.

Try to find explicit formulas for A^{2n} and A^{2n+1}. To do this, it might be worth it to evaluate analytically A^2, A^3, A^4, *etc.* This should give you an idea of what the general expression should be. Then prove your "guess" via induction.

Problem 5.5:

Set $b = \sup_x \int_0^1 |k(x,y)|\,dy$.

First we observe that

$$||Ku|| = \sup_x \Big| \int_0^1 k(x,y)u(y)dy \Big| \leq \sup_x \int_0^1 |k(x,y)|\ |u(y)|dy$$
$$\leq \sup_x \int_0^1 |k(x,y)|\ ||u||dy = b\,||u||,$$

which proves that $||K|| \leq b$. Next, prove that there exists a sequence $(u_n)_{n=1}^\infty$ of continuous functions such $|u_n(y)| \leq 1$ for all y, and

$$\lim_{n\to\infty} \int_0^1 \Big|\,|k(x,y)| - u_n(y)\Big|\,dy = 0.$$

(Prove that such a sequence exists!) Then $||u_n|| = 1$ so

$$||K|| \geq ||Ku_n|| \to b.$$

(Fill in details!)

Problem 5.7: Observe that

$$\sin(\pi(x-y)) = \sin(\pi x)\cos(\pi y) - \cos(\pi x)\sin(\pi y).$$

Consequently

$$[Kf](x) = \sin(\pi x)\int_0^1 \cos(\pi y)f(y)dy - \cos(\pi x)\int_0^1 \sin(\pi y)f(y)dy.$$

From this formula, it is not hard to prove that the range of K is the linear span of the functions $u_1(x) = \sin(\pi x)$ and $u_2(x) = \cos(\pi x)$. The kernel consists of all functions u such that

$$\int_0^1 \cos(\pi y)u(y)dy = 0, \qquad \text{and} \qquad \int_0^1 \sin(\pi y)u(y)dy = 0.$$

Problem 5.8: Review the definition of equivalent norms. Assume that two norms on S are equivalent, and then prove that the corresponding operator norms are equivalent. Then go in the other direction.