**Homework set 2 — APPM5440, Fall 2009**

From the textbook: 1.8, 1.9, 1.10, 1.12, 1.13.

**Problem 1:** Let $(X, d)$ be a metric space, and let $\Omega \subseteq X$. Prove that:

$$\Omega \text{ is dense in } X \quad \Leftrightarrow \quad \forall\, x \in X, \varepsilon > 0, \quad \exists y \in \Omega \text{ such that } x \in B_\varepsilon(y).$$

(In words: $\Omega$ is dense iff for every $x$ in $X$, and for every $\varepsilon > 0$, there exists an element $y \in \Omega$ that is within distance $\varepsilon$ of $x$.)

**Problem 2:** Suppose that $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are Cauchy sequences in a metric space $(X, d)$. Prove that the sequence $(d(x_n, y_n))_{n=1}^\infty$ converges.

**Problem 3:** The proof that every metric space has a completion that we postponed contains an important technique called the *Cantor diagonal argument.* Try to use it to prove that the real numbers are not countable. Hint: Assume that there exists an enumeration $(r^{(n)})_{n=1}^\infty$ of all real numbers in the interval $(0, 1)$. Suppose that each $r^{(n)}$ has a binary number expansion

$$r^{(n)} = 0.b_1^{(n)}\, b_2^{(n)}\, b_3^{(n)} \dots$$

(so that each $b_j^{(n)}$ is either 0 or 1) and use the "diagonal" technique to construct a real number that is not in the sequence. (There is a full solution in the Wikipedia article on Cantor's diagonal argument.)