Homework set 5 — APPM5440 — Spring 2009

2.7: Set $I = [0, 1]$, and $\Omega = \{f \in C(I) : \text{Lip}(f) \leq 1, \int f = 0\}$.

We will use the Arzelà-Ascoli theorem, of course.

The Lipschitz condition implies that Ω is equicontinuous. (To prove this, fix any $\varepsilon > 0$. Set $\delta = \varepsilon$. Then for any $f \in \Omega$, and $|x - y| < \delta$, we have $|f(x) - f(y)| \leq \text{Lip}(f)\,|x - y| \leq |x - y| < \varepsilon$.)

To prove that Ω is bounded, note that if $\int f = 0$, and f is continuous, then there must exist an $x_0 \in I$ such that $f(x_0) = 0$. Then for any $x \in I$ and any $f \in \Omega$, we have $|f(x)| = |f(x) - f(x_0)| \leq \text{Lip}(f)\,|x - x_0| \leq |x - x_0| \leq 1$. So $||f||_{\text{u}} \leq 1$.

Finally we need to prove that Ω is closed. Let (f_n) be a Cauchy sequence in Ω. Since $C(I)$ is complete, there exists an $f \in C(I)$ such that $f_n \to f$ uniformly. We need to prove that $f \in \Omega$. Since $f_n \to f$ uniformly, we know both that $\text{Lip}(f) \leq \limsup_{n\to\infty} \text{Lip}(f_n) \leq 1$, and that $\int f = \lim_{n\to\infty} \int f_n = 0$. This proves that $f \in \Omega$.

2.8: We will explicitly construct a dense countable subset Ω of $C([a,\, b])$. Without loss of generality, we can assume that $a = 0$ and that $b = 1$.

For $n = 1, 2, \dots,$ and for $j = 0, 1, 2, \dots, n$, set $x_j^{(n)} = j/n$. Let Ω_n denote the subset of $C(I)$ of functions that (1) are linear on each interval $[x_{j-1}^{(n)}, x_j^{(n)}]$, and (2) take on rational values for each $x_j^{(n)}$. Since each function in Ω_n is uniquely defined by its values on the $x_j^{(n)}$'s, we can identify Ω_n by $\mathbb{Q}^{n+1}$. Hence Ω_n is countable.

Set $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Since each Ω_n is countable, Ω is countable.

It remains to prove that Ω is dense in $C(I)$. Fix any $f \in C(I)$, and any $\varepsilon > 0$. Since I is compact, f is uniformly continuous on I so there exists a $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon/2$. Pick an n such that $1/n < \delta$, and pick a $\varphi \in \Omega_n$ such that $|\varphi(x_j^{(n)}) - f(x_j^{(n)})| < \varepsilon/2$ for $j = 0, 1, 2, \dots, n$. We will prove that $||\varphi - f||_{\mathrm{u}} < \varepsilon$: Fix an $x \in I$. Then pick $j \in \{1, 2, \dots, n\}$ so that $x \in [x_{j-1}^{(n)},\, x_j^{(n)}]$. Since φ is linear in this interval, there is a number $\alpha \in [0,\, 1]$ such that

$$\varphi(x) = \alpha\, \varphi(x_{j-1}^{(n)}) + (1 - \alpha)\, \varphi(x_j^{(n)}).$$

Now

$$\begin{aligned}(1) \quad |f(x) - \varphi(x)| &= |\alpha\, f(x) + (1-\alpha)\, f(x) - \alpha\, \varphi(x_{j-1}^{(n)}) - (1-\alpha)\, \varphi(x_j^{(n)})| \\ &\le \alpha\, |f(x) - \varphi(x_{j-1}^{(n)})| + (1-\alpha)\, |f(x) - \varphi(x_j^{(n)})|.\end{aligned}$$

Since $|f(x) - f(x_{j-1}^{(n)})| \le \varepsilon/2$ (by the uniform continuity) and since $|f(x_{j-1}^{(n)}) - \varphi(x_{j-1}^{(n)}| < \varepsilon/2$ (by the choice of φ), we have

$$(2) \qquad |f(x) - \varphi(x_{j-1}^{(n)})| \le |f(x) - f(x_{j-1}^{(n)})| + |f(x_{j-1}^{(n)}) - \varphi(x_{j-1}^{(n)}| < \varepsilon.$$

Analogously,

$$(3) \qquad |f(x) - \varphi(x_j^{(n)})| \le |f(x) - f(x_j^{(n)})| + |f(x_j^{(n)}) - \varphi(x_j^{(n)}| < \varepsilon.$$

Together, (1), (2), and (3) imply that $|f(x) - \varphi(x)| < \varepsilon$.

2.9: (a) Suppose that $w(x) > 0$ for $x \in (0, 1)$. Then $||\cdot||_w$ is a norm since:
(i) $||\lambda f||_w = \sup_x w(x)|\lambda f(x)| = |\lambda| \sup_x w(x)|f(x)| = |\lambda|\,||f||_w$.
(ii) $||f + g||_w = \sup_x w(x)|f(x) + g(x)| \leq \sup_x w(x)\big(|f(x)| + |g(x)|\big) \leq \sup_x w(x)|f(x)| + \sup_x w(x)|g(x)| = ||f||_w + ||g||_w$.
(iii) If $f = 0$, then clearly $||f||_w = 0$. Conversely, if $f \neq 0$, then $f(x_0) \neq 0$ for some $x_0 \in (0, 1)$. Then $||f||_w \geq w(x_0)|f(x_0)| > 0$.

(b) Assume that $w(x) > 0$ for $x \in [0, 1] =: I$. Set $m = \inf_{x\in I} w(x)$ and $M = \sup_{x\in I} w(x)$. Since I is compact and w is continuous, w attains both its inf and its sup, and therefore $m > 0$ and $M < \infty$. Then

$$||f||_{\mathrm{u}} = \sup_{x\in I} |f(x)| \geq \sup_{x\in I} \frac{w(x)}{M}|f(x)| = \frac{1}{M}||f||_w.$$

and

$$||f||_{\mathrm{u}} = \sup_{x\in I} |f(x)| \leq \sup_{x\in I} \frac{w(x)}{m}|f(x)| = \frac{1}{m}||f||_w.$$

It follows that

$$\frac{1}{M}||f||_w \leq ||f||_{\mathrm{u}} \leq \frac{1}{m}||f||_w.$$

(c) Set $|||f||| = \sup_{x\in I} |x\, f(x)|$. We will prove that $|||\cdot|||$ is not equivalent to the uniform norm. Set for $n = 1, 2, \dots$

$$f_n(x) = \begin{cases} 1 - nx & x \in [0, 1/n], \\ 0 & x \in (1/n, 1]. \end{cases}$$

Then $||f_n||_{\mathrm{u}} = 1$ for all n, while $|||f_n||| = \sup_x x|f_n(x)| \leq 1/n$. This proves that there cannot be a finite M such that $||f||_{\mathrm{u}} \leq M\,|||f|||_w$ for all f.

(d) We will prove that the set $C(I)$ equipped with the norm $|||\cdot|||$ is not a Banach space by constructing a Cauchy sequence with no limit point in $C(I)$. For $n = 1, 2, \dots,$ define $f_n \in C(I)$ by

$$f_n(x) = \begin{cases} x^{-1/2} & x \in (1/n, 1], \\ \sqrt{n} & x \in [0, 1/n]. \end{cases}$$

Fix a positive integer N. Then, if $m, n \geq N$, we have

$$\begin{aligned} |||f_n - f_m||| &= \sup_{x\in[0,1/N]} x\,|f_n(x) - f_m(x)| \\ &\leq \sup_{x\in[0,1/N]} \big(x|f_n(x)| + x|f_m(x)|\big) \\ &\leq \sup_{x\in(0,1/N)} \big(x \cdot x^{-1/2} + x \cdot x^{-1/2}\big) = 2N^{-1/2}. \end{aligned}$$

Consequently, $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence. But f_n cannot converge uniformly to any function in $C(I)$. (To prove the last contention, suppose that $f_n \to f$ for some $f \in C(I)$. Then $f(0) = \lim_{n\to\infty} f_n(0) = \infty$, which is a contradiction.)