Applied Analysis (APPM 5440): Section exam 1

8:30am – 9:50am, Sep. 21, 2009. Closed books.

The following problems are worth 20 points each.

Problem 1: In what follows, (X, d_X) and (Y, d_Y) are metric spaces.

(a) Define what it means for a function $f : X \to Y$ to be continuous.

(b) Define what it means for a subset Ω of X to be compact.

(c) Define what a completion of (X, d_X) is.

(d) Let Ω be a subset of X. Define the closure of Ω.

Problem 2: Let $\mathbb{Q}$ denote the set of rational numbers. On $\mathbb{Q}$, we define the discrete metric

$$d(x, y) = \begin{cases} 0, & \text{when } x = y, \\ 1, & \text{when } x \neq y. \end{cases}$$

(a) What subsets of $\mathbb{Q}$ are open in $(\mathbb{Q}, d)$? Prove your claim.

(b) Specify which sequences in $(\mathbb{Q}, d)$ are convergent. *No motivation required.*

(c) Set $\Omega = \{q \in \mathbb{Q} : q^2 < 2\}$. What is the closure of Ω in $(\mathbb{Q}, d)$? *No motivation required.*

(d) Set $\Omega = \{q \in \mathbb{Q} : q^2 < 2\}$. What is the completion of (Ω, d)? *No motivation required.*

Problem 3: Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Set $y_n = x_{2n}$. Which of the following two statements must necessarily be true:

$$\begin{array}{ll} \text{(a)} & \limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n, \\ \text{(b)} & \limsup_{n\to\infty} y_n \leq \limsup_{n\to\infty} x_n. \end{array}$$

Motivate your answers carefully. State the definition of "limsup" that you use and make sure that your argument follows directly from this definition.

Problem 4: Let $(X, ||\cdot||)$ be a normed linear space. Suppose that every sequence $(x_n)_{n=1}^{\infty}$ in X such that $||x_m - x_n|| \leq 1/N$ whenever $m, n \geq N$ converges to a point in X. Is $(X, ||\cdot||)$ necessarily complete? Prove this if you answer yes, and give a counterexample if you answer no.

Problem 5: Let (X, d) be a compact metric space. Let $C_{\text{b}}(X)$ denote the set of all bounded real-valued continuous functions on X, equipped with the uniform norm,

$$||f||_{\text{u}} = \sup_{x \in X} |f(x)|.$$

Prove that $C_{\text{b}}(X)$ is complete.