Applied Analysis (APPM 5440): Section exam 3
8:30am – 9:50am, Nov. 30, 2009. Closed books.

Problem 1: (24p) With X a Banach space, which statements are necessarily true (please motivate):

(a) If $S, T \in \mathcal{B}(X)$ and T is compact, then ST is compact.

(b) If $S, T \in \mathcal{B}(X)$ and T is compact, then TS is compact.

(c) Suppose that for $n = 1, 2, 3, \dots$, we know that $T_n \in \mathcal{B}(X)$ has finite dimensional range, and that there exists an operator $T \in \mathcal{B}(X)$ such that T_n converges strongly to T. Then T is compact.

Solution:

Definition of a compact operator that we use: T is compact if (Tx_n) has a bounded subsequence whenever (x_n) is a bounded sequence.

(a) TRUE. Let (x_n) be a bounded sequence. Since T is compact, we can pick a convergent subsequence (Tx_{n_j}) of (Tx_n). Since S is bounded, it is also continuous, and therefore (STx_{n_j}) is convergent.

(b) TRUE. Let (x_n) be a bounded sequence. Set $y_n = Sx_n$. Since S is bounded, (y_n) is bounded. Since T is compact, we can pick a subsequence (y_{n_j}) such that (Ty_{n_j}) is convergent. Now simply observe that $Ty_{n_j} = TSx_{n_j}$.

(c) FALSE. Consider $X = \ell^2$ and T_n defined by

$$T_n x = (x_1, x_2, \dots, x_n, 0, 0, \dots).$$

The dimension of the range of T_n is n. For any fixed x, we have

$$||T_n x - x|| = \left(\sum_{j=n+1}^{\infty} |x_j|^2 \right)^{1/2} \to 0, \qquad \text{as } n \to \infty.$$

Consequently, $T_n x \to x$ for any fixed x, which is to say that T_n converges strongly to the identity operator. The identity operator is not compact, so this provides a counterexample.

Problem 2: (28p) Set $X = \ell^3$. Define the operator $T \in \mathcal{B}(X)$ via

$$T\,(x_1,\, x_2,\, x_3,\, \dots) = (\tfrac{1}{1}x_1,\, \tfrac{1}{2}x_2,\, \tfrac{1}{3}x_3,\, \dots).$$

Which of the following statements are necessarily true? Motivate your answers.

(a) $\text{Ran}(T)$ is a linear subspace.

(b) $\text{Ker}(T)$ is a linear subspace.

(c) $\text{Ran}(T)$ is topologically closed.

(d) $\text{Ker}(T)$ is topologically closed.

Solution:

(a) TRUE. Suppose $y_1, y_2 \in \text{Ran}(T)$. Then there exist $x_1, x_2 \in X$ such that $y_1 = Tx_1$ and $y_2 = Tx_2$. Now for any $r_1, r_2 \in \mathbb{R}$,

$$r_1 y_1 + r_2 y_2 = r_1 T x_1 + r_2 T x_2 = T(r_1 x_1 + r_2 x_2).$$

Since $r_1 x_1 + r_2 x_2 \in X$, it must be that $r_1 y_1 + r_2 y_2 \in \text{Ran}(T)$.

(b) TRUE. Suppose that $x_1, x_2 \in \text{Ker}(T)$ and $r_1, r_2 \in \mathbb{R}$. Then

$$T(r_1 x_1 + r_2 x_2) = r_1 T x_1 + r_2 T x_2 = r_1 \cdot 0 + r_2 \cdot 0 = 0,$$

so $r_1 x_1 + r_2 x_2 \in \text{Ker}(T)$.

(c) FALSE. We will show that the vector $y = (1,\, 1/2,\, 1/3,\, 1/4,\, \dots)$ belongs to the closure of $\text{Ran}(T)$ but not to $\text{Ran}(T)$.

To see that $y \in \overline{\text{Ran}(T)}$, set $x_n = \sum_{j=1}^{n} e_j = (1,\, 1,\, \dots,\, 1,\, 0,\, 0,\, \dots)$, and set $y_n = Tx_n = \sum_{j=1}^{n} \frac{1}{j} e_j = (1,\, 1/2,\, 1/3,\, \dots,\, 1/n,\, 0,\, 0,\, \dots)$. Since $x_n \in X$, we clearly have $y_n \in \text{Ran}(T)$, and, moreover,

$$||y_n - y|| = \left(\sum_{j=n+1}^{\infty} \frac{1}{j^3} \right)^{1/3} \to 0, \qquad \text{as } n \to \infty.$$

This proves that y belongs to the closure of the range.

To see that y cannot belong to the range itself, suppose that there is an element $x \in X$ such that $Tx = y$. Looking elementwise, we then see that every entry of x would have to be one, which is impossible since then x would have infinite norm.

(d) TRUE. We have $||T|| \leq 1$ so T is continuous. Therefore, if $x_n \in \text{Ker}(T)$, and $x_n \to x$, we have

$$Tx = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} 0 = 0.$$

Alternate solution for (b) and (d): Observe that $\text{Ker}(T) = \{0\}$. The set $\{0\}$ is obviously a both a linear subspace, and a closed set.

Problem 3: (24p) Set $X = \ell^\infty$. Define for any positive integer n a linear map φ_n from X to $\mathbb{R}$ via

$$\varphi_n(x) = \frac{1}{n}\sum_{j=1}^{n} x_j.$$

(a) Prove that φ_n is bounded and determine its norm.

(b) Does $(\varphi_n)_{n=1}^\infty$ converge in norm in X^*?

(c) Does $(\varphi_n)_{n=1}^\infty$ converge weakly in X^*?

For 6 points extra credit: Answer (a), (b), and (c), again, but now for the space $X = \ell^1$.

Solution:

(a) First we prove that $||\varphi_n|| \leq 1$:

$$|\varphi_n(x)| = \left|\frac{1}{n}\sum_{j=1}^{n} x_j\right| \leq \frac{1}{n}\sum_{j=1}^{n}|x_j| \leq \frac{1}{n}\sum_{j=1}^{n}||x|| = ||x||.$$

Next we prove that $||\varphi_n|| \geq 1$. To this end, set $x_n = \sum_{j=1}^{n} e_j$ where e_j are the canonical unit vectors. (In other words, x_n is the vector consisting of n ones, and then all zeros.) Then since $||x_n|| = 1$,

$$||\varphi_n|| = \sup_{||x||=1} |\varphi_n(x)| \geq \sup_n |\varphi_n(x_n)| = \sup_n \frac{1}{n}\sum_{j=1}^{n} 1 = 1.$$

It follows that $||\varphi_n|| = 1$.

(c) We will construct an $F \in X^{**}$ such that $(F(\varphi_n))_{n=1}^\infty$ does not converge. This shows that (φ_n) does not converge weakly. Our F takes that form $F(\varphi) = \varphi(x)$ for the particular $x \in X$ defined by

$$x = \sum_{n=1}^{\infty}\sum_{j=2^{n-1}+1}^{2^n} (-1)^j\, e_j = [0,\, -1,\, 1\, 1,\, -1,\, -1,\, -1,\, -1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, -1,\, \dots]$$

Then

$$F(\varphi_{2^n}) = \begin{cases} -\dfrac{2^n+1}{3\cdot 2^n} & \text{when } n \text{ is odd,} \\[2ex] \dfrac{2^n-1}{3\cdot 2^n} & \text{when } n \text{ is even.} \end{cases}$$

Since $F(\varphi_{2^n}) \to -1/3$ for n odd, and $F(\varphi_{2^n}) \to 1/3$ for n even, $(F(\varphi_{2^n}))_{n=1}^\infty$ cannot converge.

(The above is perhaps a little obtuse but the idea is simple: φ_n is the averaging operator. The sequence of averaging operators cannot converge since the averages of a bounded sequence need not converge. The given x is just a particular choice of a sequence whose average does not converge.)

(b) Since (φ_n) does not converge weakly, it certainly does not converge in norm.

Extra credit: First we prove that $||\varphi_n|| \leq 1/n$:

$$|\varphi(x)| = \left|\frac{1}{n}\sum_{j=1}^{n} x_j\right| \leq \frac{1}{n}\sum_{j=1}^{n}|x_j| = \frac{1}{n}||x||.$$

To see that $||\varphi_n|| \geq 1/n$ simply use the same x_n as in part (b) above. Then $|\varphi_n(x_n)| = 1 = (1/n)||x_n||$.

Having established that $||\varphi_n|| = 1/n$, it is obvious that $\varphi_n \to 0$ in norm, and therefore that φ_n also converges to zero weakly.

Problem 4: (24p) Let X be a topological space that satisfies the Hausdorff property. Let K be a compact subset of X.

(a) State the definition of the Hausdorff property.

(b) State the definition of a compact set in a general topological space.

(c) Prove that K is necessarily closed.

Solution:

(a) For any $x, y \in X$ such that $x \neq y$, there exist $G, H \in \mathcal{T}$ such that $x \in G$, $y \in H$, and $G \cap H = \emptyset$.

(b) Any open cover $(G_\alpha)_{\alpha \in A}$ of K has a finite subcover $(G_{\alpha_j})_{j=1}^{J}$.

(c) Suppose that $x \in K^{\rm c}$. We will construct an open set G such that $x \in G \subseteq K^{\rm c}$. This proves that $K^{\rm c}$ is open, which is to say that K is closed.

For any $y \in K$, we invoke the Hausdorff property to assert the existence of disjoint open sets G_y and H_y such that $y \in H_y$ and $x \in G_y$. Now observe that

$$K = \bigcup_{y \in K}\{y\} \subseteq \bigcup_{y \in K} H_y.$$

So $\{H_y\}_{y \in K}$ is an open cover of K. Since K is compact, we can pick a finite subcover

$$K \subseteq \bigcup_{j=1}^{J} H_{y_j}.$$

Now set

$$G = \bigcap_{j=1}^{J} G_{y_j}.$$

Since G_{y_j} and H_{y_j} are disjoint, $G_{y_j} \subseteq H_{y_j}^{\rm c}$, and therefore

$$G = \bigcap_{j=1}^{J} G_{y_j} \subseteq \bigcap_{j=1}^{J} H_{y_j}^{\rm c} = \left(\bigcup_{j=1}^{J} H_{y_j}\right)^{\rm c} \subseteq K^{\rm c}.$$

Finally note that $x \in G$, and that since G is a finite intersection of open sets, it must itself be open.