Applied Analysis (APPM 5450): Final Exam
7.30am – 10.00am, May 9, 2006. Closed books.

Notes: No motivation is required for problems 1 and 2. For problems 3, 4, and 5, please motivate your answers clearly, but briefly. Statements given as theorems, lemmas, *etc.*, in the class material may be used without justification. The use of results from homework problems without justification will result in a loss of points. Integral signs denote Lebesgue integrals unless otherwise stated (in this exam, as well as in your solutions).

Problem 1: No motivation required. Two points each.

(a) Let H be a Hilbert space and let $A \in \mathcal{B}(H)$. Define what it means for A to be an "orthogonal projection".

(b) Let H denote a Hilbert space, let A denote a compact self-adjoint operator on H, and let $(\varphi_n)_{n=1}^{\infty}$ denote an orthonormal basis on H such that $A\,\varphi_n = \lambda_n\,\varphi_n$ for some numbers λ_n. Which statements must be true?

(1) $\lim_{n\to\infty} \lambda_n = 0$.
(2) If c is a non-zero real number, then the number of λ_n's that equal c is finite. (And yes, "zero" is a finite number.)
(3) At least one λ_n must equal zero.

(c) Define convergence on the Schwartz space $\mathcal{S}(\mathbb{R})$.

(d) Consider the function $f:\ \mathbb{R}^2 \to \mathbb{R}:\ (x_1, x_2) \mapsto x_2$. What is the Fourier transform (in the distributional sense) of f? (Don't worry about constants.)

(e) Let $\mathcal{F}$ denote the Fourier transform. Which statements are true?

(1) $\mathcal{F}[L^1(\mathbb{R})] \subseteq L^1(\mathbb{R})$
(2) $\mathcal{F}[L^2(\mathbb{R})] = L^2(\mathbb{R})$
(3) $\mathcal{F}[S^*(\mathbb{R})] = S^*(\mathbb{R})$

(f) State Fatou's lemma.

(g) Set $I = [0,1]$, fix a function $\psi \in C(I)$, and consider the map $f:\ C(I) \to C(I)$ that is defined by

$$f(\varphi):\ x \mapsto \psi(x) \int_0^1 \sin(\varphi(y))\,dy.$$

Calculate the (Fréchet) derivative of f. (Give a formula for the derivative, and specify explicitly what kind of object it is.)

Problem 2: Consider the operator A on the Hilbert space $H = L^2([-\pi, \pi])$ that is defined by

$$[Au](x) = \alpha \int_{-\pi}^{\pi} \cos(2x - 2y)\, u(y)\, dy,$$

where α is a given complex number. For the following questions, no motivation is required.

(a) Determine $\mathrm{ran}(A)$. (2p)

(b) Determine $\sigma(A)$ and classify each point in the spectrum as belonging to either $\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{c}}(A)$, or $\sigma_{\mathrm{r}}(A)$. (2p)

(c) For what numbers α, if any, is A a self-adjoint operator? (1p)

(d) For what numbers α, if any, is A a unitary operator? (1p)

(e) For what numbers α, if any, is A a projection? (1p)

Problem 3: Let $\mathcal{S}(\mathbb{R})$ denote the set of Schwartz functions on the line. Consider the map

$$T:\ \mathcal{S}(\mathbb{R}) \to \mathbb{R}:\ \varphi \mapsto \lim_{\varepsilon \searrow 0} \int_{|x| \geq \varepsilon} \frac{1}{x}\, \varphi(x)\, dx.$$

Prove that T is continuous. (5p)

Problem 4: Let f_n be a sequence of measurable functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}$. Let g be a function in $L^1(\mathbb{R})$. Determine

$$\lim_{n\to\infty} \int_0^\infty \frac{f_n(x)\, g(x)}{1 + (f_n(x))^4}\, dx.$$

Justify your answer. (5p)

Problem 5: Let p be a real number such that $1 \leq p < \infty$, and $p \neq 2$. Prove that $L^p(\mathbb{R}^d)$ is not a Hilbert space. (In other words, prove that the norm on L^p cannot be generated by an inner product.) (4p)