Homework set 2 — APPM5450

Problem 1: Let H be a Hilbert space, and let $(\varphi_n)_{n=1}^{\infty}$ denote an orthonormal basis for H. Given a bounded sequence of complex number $(\lambda_n)_{n=1}^{\infty}$, define the operator A by setting $A\,u = \sum_{n=1}^{\infty} \lambda_n\,\varphi_n\,(\varphi_n,\,u)$.

(a) Prove that $||A|| = \sup_n |\lambda_n|$.

(b) Prove that $A^{*}u = \sum_{n=1}^{\infty} \bar{\lambda}_n \varphi_n\,(\varphi_n,\,u)$. Conclude that A is self-adjoint iff all λ_n's are real. When is A skew-symmetric?

Problem 2: Prove that if P is a projection on a Hilbert space H, then the following three statements are equivalent:
(1) P is orthogonal (*i.e.* $\ker(P) = \text{ran}(P)^{\perp}$).
(2) P is self-adjoint.
(3) $||P|| = 1$.

Problem 3: Set $H = l^2(\mathbb{Z})$ and let R denote the right-shift operator (so that if $y = R\,x$, then $y_n = x_{n-1}$). Construct R^{*}. Prove that R is unitary.

Problem 4: Consider the Hilbert space $L^2(\mathbb{T})$. Let k denote a continuous function on $\mathbb{T}^2$ that takes on complex values. Let A denote the operator $[Au](x) = \int_{\mathbb{T}} k(x,y)\,u(y)\,dy$. Prove that $[A^{*}u](x) = \int_{\mathbb{T}} \overline{k(y,x)}\,u(y)\,dy$. Conclude that A is self-adjoint iff $k(x,y) = \overline{k(y,x)}$ $\forall x, y \in \mathbb{T}$.

Problem 5: Assume that you have proved that in a Hilbert space, $\text{ran}(I - K)$ is closed whenever K has finite rank. Prove from this that $\text{ran}(I - K)$ is closed for all compact operators K. (Then if you feel industrious, feel free to try proving the statement given in the first sentence.)

From the textbook: 8.5.