Homework set 3 — APPM5450, Spring 2006

Problem 1: Consider the Hilbert space $H = l^2(\mathbb{N})$; let e_n denote the canonical basis elements. Which of the following sequences converge weakly? Which have convergent subsequences?
(a) $x_n = n\, e_n$.
(b) $x_n = n^{-1/2} \sum_{j=1}^{n} e_j$.
(c) $x_n = e_n + e_m$ where $m = \text{mod}(n, 2)$.

Problem 2: Consider the Hilbert space $H = L^2(\mathbb{T})$, and the sequence of functions $\varphi_n(x) = x^2 \sin(nx)$. Does $(\varphi_n)_{n=1}^{\infty}$ converge weakly in H? If so, what does it converge to?

Problem 3: Let A be a self-adjoint operator on a Hilbert space H. Let u be an element of H, and set $u_n = e^{i\,n\,A}\,u$. Prove that $(u_n)_{n=1}^{\infty}$ has a weakly convergent subsequence.

Problem 4: Consider the Hilbert space $X = L^2(\mathbb{T})$, and let $\mathcal{P}$ denote the space of trigonometric polynomials (which is dense in X). For $u \in \mathcal{P}$, let A denote the operator $[Au](x) = 100\,u(x) - 18\,u''(x) + u''''(x)$. Prove that

$$\sup_{u\in\mathcal{P},\ ||u||=1} (A\,u,\, u) = \infty.$$

Conclude that A cannot be extended to a bounded linear operator on X. Prove that for $u, v \in \mathcal{P}$, it is the case that $(A\,u,\, v) = (u,\, A\,v)$. Determine

$$\inf_{u\in\mathcal{P},\ ||u||=1} (A\,u,\, u).$$

Prove that

$$a(u,\, v) = (A\,u,\, v)$$

is a bilinear form on $\mathcal{P}$. Prove that the norm $||\cdot||_A$ defined by

$$||u||_A = \sqrt{a(u, u)}$$

is equivalent to the norm

$$||u||_{H^2} = \sqrt{||u||^2 + ||u''||^2}.$$

Conclude that the closure of $\mathcal{P}$ under the norm $||\cdot||_A$ is $H^2(\mathbb{T})$ (as defined in Section 7.2).