Homework set 7 — APPM5450

Problem 1: Define $f \in \mathcal{S}^*$ by $f(x) = |x|$. Compute f' and f''.

Problem 2: Prove that if $f \in C^\infty(\mathbb{R}^d)$, and for every $\alpha \in \mathbb{Z}^d$, there exist finite C and N such that $|\partial^\alpha f(x)| \leq C(1 + |x|^N)$, then $f\varphi \in \mathcal{S}$ whenever $\varphi \in \mathcal{S}$. Moreover, prove that if $\varphi_n \to \varphi$, then $f\varphi_n \to f\varphi$.

Problem 3: Let $\mathcal{D}$ denote the linear space $C_{\mathrm{c}}^\infty(\mathbb{R}^d)$. We define a topology on $\mathcal{D}$ by saying that $\varphi_n \to \varphi$ if and only if there exists a compact set $K \subseteq \mathbb{R}^d$ such that $\mathrm{supp}(\varphi_n) \subseteq K$ for all n, and $||\partial^\alpha \varphi_n - \partial^\alpha \varphi||_{\mathrm{u}} \to 0$ for all $\alpha \in \mathbb{Z}_+^d$.
(a) Prove that $\mathcal{D}$ is a linear subspace of $\mathcal{S}$.
(b) Prove that the set $\mathcal{D}$ is not closed in the topology of $\mathcal{S}$.
(c) Prove that if $\varphi_n \to \varphi$ in $\mathcal{D}$, then $\varphi_n \to \varphi$ in $\mathcal{S}$.

From the textbook: 11.5, 11.6, 11.7, 11.8.