# The Implicit and Inverse Function Theorems

Notes to supplement Chapter 13.

**Remark:** These notes are still in draft form. Examples will be added to Section 5. If you see any errors, please let me know.

## 1. Notation

Let $X$ and $Y$ be two normed linear spaces, and let $f\colon X \to Y$ be a function defined in some neighborhood of the origin of $X$. We say that $f(x) = o(||x||^n)$ if

$$\lim_{x\to 0} \frac{||f(x)||_Y}{||x||_X^n} = 0.$$

Analogously, we say that $f(x) = O(||x||^n)$ if there exists some number $C$, and some neighborhood $G$ of the origin in $X$ such that

$$||f(x)||_Y \leq C\,||x||_X^n, \qquad \forall x \in G.$$

## 2. Differentiation on Banach Space

Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is "differentiable" at a point $x_0$ if there exists a number $a$ such that

$$f(x) = f(x_0) + a\,(x - x_0) + o(|x - x_0|). \tag{1}$$

We usually write $a = f'(x)$. The right hand side of (1) is a linear approximation of $f$, valid near $x_0$. This definition can straight-forwardly be generalized to functions between two Banach spaces.

**Definition 1.** Let $X$ and $Y$ be Banach spaces, let $f$ be a map from $X$ to $Y$, and let $x_0$ be a point in $X$. We say that $f$ is *differentiable* at $x_0$ if there exists a map $A \in \mathcal{B}(X,Y)$ such that

$$f(x) = f(x_0) + A\,(x - x_0) + o(||x - x_0||).$$

The number $A$ is called the "Fréchet Derivative" of $f$ at $x_0$. We write $A = f'(x) = df = D\,f = f_x$.

Note that the definition makes sense even if $f$ is defined only in a neighborhood of $x_0$ (it does not need to be defined on all of $x$).

It follows directly from the definition that if $f$ is differentiable at $x_0$, then it is also continuous as $x_0$.

The function $f'(x)$ is **not** a map from $X$ to $Y$, it is a map from $X$ to $\mathcal{B}(X,Y)$.

**Example 1:** Let $f$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let $f_i$ denote the component functions of $f$ so that $f = (f_1,\, f_2, \dots, f_m)$. If the partial derivatives

$$f_{i,j} = \frac{\partial f_i}{\partial x_j}$$

all exist at some $x_0 \in \mathbb{R}^n$, then $f$ is differentiable at $x_0$ and

$$f'(x_0) = \begin{bmatrix} f_{1,1}(x_0) & f_{1,2}(x_0) & \cdots & f_{1,n}(x_0) \\ f_{2,1}(x_0) & f_{2,2}(x_0) & \cdots & f_{2,n}(x_0) \\ \vdots & \vdots & & \vdots \\ f_{m,1}(x_0) & f_{m,2}(x_0) & \cdots & f_{m,n}(x_0) \end{bmatrix}.$$

**Example:** Let $(X, \mu)$ be a measure space and consider the function

$$f : L^3(X, \mu) \to L^1(X, \mu) : \varphi \mapsto \varphi^3.$$

In order to see if $f$ is differentiable, we need to see if there exists an $A \in \mathcal{B}(L^3, L^1)$ such that

$$\lim_{||\psi||_3 \to 0} \frac{||f(\varphi + \psi) - f(\varphi) - A\,\psi||_1}{||\psi||_3} = 0. \tag{2}$$

We have

$$f(\varphi + \psi) = \varphi^3 + 3\varphi^2\,\psi + 3\varphi\,\psi^2 + \psi^3.$$

Therefore, if $f$ is differentiable, we must have $A\,\psi = 3\varphi^2\,\psi$. That $A$ is a bounded map is clear since (applying Hölder's inequality with $p = 3$, $q = 2/3$)

$$||A\,\psi||_1 = 3\int |\varphi|^2\,|\psi| \leq 3\,\left(\int |\psi|^3\right)^{1/3}\,\left(\int |\varphi|^3\right)^{2/3} = 3\,||\psi||_3\,||\varphi||_3^2$$

A similar calculation shows that

$$\begin{aligned} ||f(\varphi + \psi) - f(\varphi) - A\,\psi||_1 &= ||3\,\varphi\,\psi^2 + \psi^3||_1 \\ &\leq 3\,||\varphi\,\psi^2||_1 + ||\psi^3||_1 \leq 3\,||\varphi||_3\,||\psi||_3^2 + ||\psi||_3^3. \end{aligned}$$

It follows that (2) holds. Thus $f$ is differentiable, and $f'(\varphi) : \psi \mapsto 3\,\varphi^2\,\psi$.

**Example:** Set $I = [0,\,1]$ and consider the function

$$f : C(I) \to \mathbb{R} : \varphi \mapsto \int_0^1 \sin(\varphi(x))\,dx.$$

The function $f$ is differentiable at $\varphi$ if there exists an $A \in \mathcal{B}(C(I), R) = C(I)^*$ such that

$$\lim_{||\psi||_{\mathrm{u}} \to 0} \frac{|f(\varphi + \psi) - A\,\psi|}{||\psi||_{\mathrm{u}}} = 0. \tag{3}$$

We find that

$$f(\varphi + \psi) = \int \sin(\varphi + \psi) = \int (\sin\varphi\cos\psi + \cos\varphi\sin\psi).$$

When $||\psi||_{\mathrm{u}}$ is small, $\psi(x)$ is small for every $x$, and so $\cos\psi = 1 + O(\psi^2)$, and $\sin\psi = \psi + O(\psi^2)$. An informal calculation then yields

$$f(\varphi+\psi) = \int ((\sin\varphi)(1+O(\psi^2))+(\cos\varphi)(\psi+O(\psi^2))) = f(\varphi)+\int ((\cos\varphi)\psi)+O(\psi^2).$$

If $f$ is differentiable, we must have

$$A:\ \psi \mapsto \int_0^1 \cos(\varphi(x))\,\psi(x)\,dx.$$

It remains to prove that (3) holds. We have

$$\begin{aligned} f(\varphi+\psi) - f(\varphi) - A\,\psi &= \int_0^1 \big(\sin(\varphi+\psi) - \sin(\varphi) - \cos(\varphi)\,\psi\big)\,dx \\ &= \int_0^1 (\sin(\varphi)(\cos(\psi)-1) + \cos(\varphi)(\sin(\psi)-\psi))\,dx. \end{aligned}$$

Using that $|1-\cos t| \le t^2$ and $|\sin t - t| \le t^2$ for all $t \in \mathbb{R}$, we obtain

$$|f(\varphi+\psi) - f(\varphi) - A\,\psi| \le \int_0^1 (|\sin\varphi(x)| + |\cos\varphi(x)|)\,|\psi(x)|^2\,dx \le 2\,||\psi||_{\mathrm{u}}^2.$$

Therefore (3) holds, and $f$ is differentiable.

**Example:** Read Example 13.7 in the textbook.

**Theorem 1** (Chain rule)**.** *Let $X$, $Y$, and $Z$ denote Banach spaces. Suppose that the functions $f: X \to Y$, and $g: Y \to Z$ are differentiable. Then $g \circ f: X \to Z$ is differentiable, and*

$$(g \circ f)'(x) = g'(f(x))\,f'(x).$$

Note that all properties of the functions in Theorem 1 are local, so it would have been sufficient to assume only that $f$ is differentiable in some neighborhood of $x$, and $g$ is differentiable in some neighborhood of $f(x)$.

The notion of differentiation defined in Def. 1 is the direct generalization of the “Jacobian matrix” of multivariate analysis. We can also define a generalization of the concept of a directional derivative:

**Definition 2.** Let $X$ and $Y$ be Banach spaces and let $f$ denote a function from $X$ to $Y$. Letting $x$ and $u$ denote elements of $X$, we define the *directional derivative of $f$ at $x$, in the direction $u$*, by

$$(D_u f)(x) = \lim_{t\to 0} \frac{f(x+t\,u) - f(x)}{t}.$$

Note that $(D_u f)(x)$ is simply an element of $Y$.

**Remark 1.** In the environment of Banach spaces, the directional derivative is frequently called a “Gâteaux derivative”. Sometimes, this term is used to denote the vector $(D_u f)(x)$, but the text book uses a different terminology. To avoid confusion, we will avoid the term “Gâteaux derivative”.

**Example:** Let $f$ be as in Example 1, and let $u \in \mathbb{R}^m$. Then

$$(D_u f)(x) = \begin{bmatrix} u \cdot \nabla f_1(x) \\ u \cdot \nabla f_2(x) \\ \vdots \\ u \cdot \nabla f_n(x) \end{bmatrix}.$$

Note that if $X$ and $Y$ are Banach spaces, and $f$ is a differentiable function from $X$ to $Y$, then

$$f(x + t\,u) = f(x) + f'(x)\,(t\,u) + o(||t\,u||).$$

Consequently,

$$(D_u f)(x) = f'(x)\,u.$$

In other words, every (Fréchet) differentiable functions has directional derivatives in all directions. The converse is not true. In fact, it is not even true in $\mathbb{R}^2$ as the following example shows:

**Example:** Set $X = \mathbb{R}^2$ and $Y = \mathbb{R}$. Define $\Omega = \{(x_1,\, x_2) \in \mathbb{R}^2 :\, x_1^2 < x_2 < 2\,x_1^2\}$ and set $f = \chi_\Omega$. Then $(D_u\, f)(0)$ exists for all $u \in \mathbb{R}^2$, but $f$ is not even continuous at 0.

## 3. Partial derivatives

Let $X$, $Y$, and $Z$ be Banach spaces and let $F$ be a map from $X \times Y$ to $Z$. Then $F$ is differentiable at $(\hat{x}, \hat{y}) \in X \times Y$ if and only if there exist maps $A \in \mathcal{B}(X, Z)$ and $B \in \mathcal{B}(Y, Z)$ such that

$$F(x, y) = F(\hat{x}, \hat{y}) + A\,(x - \hat{x}) + B\,(y - \hat{y}) + o(||x - \hat{x}||_X + ||y - \hat{y}||_Y).$$

We call the maps $A$ and $B$ the partial derivatives of $F$ with respect to $x$ and $y$,

$$A = F_x(\hat{x}, \hat{y}), \qquad B = F_y(\hat{x}, \hat{y}).$$

Note that $A$ is simply the derivative of the map $x \mapsto F(x, \hat{y})$, and similarly $B$ is the derivative of the map $y \mapsto F(\hat{x}, y)$.

## 4. Minimization of functionals

(View this section as a large example.)

The directional derivative can be used to derive necessary conditions for a stationary point of a function. As an example, set $I = [0, 1]$, $X = C_0^1(I)$, and let us consider the functional

$$I :\, X \to \mathbb{R} :\, u \mapsto \int_0^1 L(x, u(x), u'(x))\,dx, \tag{4}$$

where $L = L(x, u, v)$ is a function that is continuously differentiable in each of its arguments. Suppose that $u \in X$ is a point where $I$ is minimized. Then for any $\varphi \in X$,

$$0 = \frac{d}{d\varepsilon} I(u + \varepsilon\varphi) = [D_\varphi I](u).$$

In other words, if $u$ is a minimizer, then the directional derivative $[D_\varphi I](u)$ must be zero for all $\varphi \in X$. For the particular functional $I$, we find that

$$\frac{d}{d\varepsilon} I(u+\varepsilon\varphi)\Big|_{\varepsilon=0} = \int_0^1 \left[L_u(x,u,u')\,\varphi + L_v(x,u,u')\,\varphi'\right]\,dx.$$

Performing a partial integration (using that $\varphi(0) = \varphi(1) = 0$), we obtain

$$0 = \int_0^1 \left[L_u(x,u,u') - \frac{d}{dx} L_v(x,u,u')\right] \varphi(x)\,dx. \tag{5}$$

For (5) to hold for every $\varphi \in C_0^1([0,1])$ we must have

$$L_u(x,u(x),u(x)') - \frac{d}{dx} L_v(x,u(x),u'(x)) = 0, \quad x \in [0,1]. \tag{6}$$

That the (potentially non-linear) ODE (6) holds is a necessary condition that a minimizer $u$ must satisfy. This equation is called the "Euler-Lagrange" equation. (The function $L$ is called the "Lagrangian" of the functional $I$.)

**Example:** Read example 13.35 in the text book.

**Example:** Consider a particle with mass $m$ moving in a potential field $\phi$. At time $t$, its position in $\mathbb{R}^d$ is $u(t)$. The Lagrangian is the difference in kinetic and potential energy,

$$L(t,u,\dot{u}) = \frac{1}{2}\,m\,|\dot{u}(t)|^2 - \phi(u(t)).$$

In other words,

$$L(t,u,v) = \frac{1}{2}\,m\,|v|^2 - \phi(u).$$

The Euler-Lagrange equations then read

$$-\phi'(u) - m\,\ddot{u} = 0,$$

which we recognize as Newton's second law.

**Example:** Consider a pendulum of mass $m$ and length $L$. Letting $\phi$ denote the angle between the vertical line, and the pendulum, we find that

$$E_{\text{kinetic}} = \frac{1}{2} m\,(L\dot{\phi})^2, \qquad E_{\text{potential}} = mgL\,(1-\cos\phi),$$

where $g$ is the gravitational constant. Thus

$$L(\phi,\dot{\phi}) = \frac{1}{2}\,mL^2\dot{\phi}^2 - mgL\,(1-\cos\phi),$$

and the Euler-Lagrange equations take the form, *cf.* (11),

$$\ddot{\phi} + \frac{g}{L}\sin\phi = 0.$$

## 5. The implicit function theorem in $\mathbb{R}^n \times \mathbb{R}$ (review)

Let $F(x, y)$ be a function that maps $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}$. The implicit function theorem gives sufficient conditions for when a level set of $F$ can be parameterized by a function $y = f(x)$.

**Theorem 2** (Implicit function theorem). *Consider a continuously differentiable function $F : \Omega \times \mathbb{R} \to \mathbb{R}$, where $\Omega$ is a open subset of $\mathbb{R}^n$. We write $F = F(x, y)$, for $x \in \Omega$, and $y \in \mathbb{R}$. Fix a point $(\hat{x}, \hat{y}) \in \Omega \times \mathbb{R}$. If $F_y(\hat{x}, \hat{y}) \neq 0$, then there exists an open set $G$ such that $\hat{x} \in G \subseteq \Omega$, and a function $f : G \to \mathbb{R}$, such that $f(\hat{x}) = \hat{y}$, and $F(x, f(x)) = F(\hat{x}, \hat{y})$ for all $x \in G$. Moreover, for $j = 1, \dots, n$, and $x \in G$,*

$$\frac{\partial f}{\partial x_j}(x) = -\frac{1}{F_y(x, f(x))}\,\frac{\partial F}{\partial x_j}(x, f(x)). \tag{7}$$

We will not prove the theorem, but note that (7) follows trivially from the chain rule: A simple differentiation with respect to $x_j$ yields:

$$F(x, f(x)) = \text{const}, \quad \Rightarrow \quad F_j(x, f(x)) + f_j(x)\, F_y(x, f(x)) = 0,$$

where the subscript $j$ refers to partial differentiation with respect to $x_j$.

For the case $n = 1$, the implicit function theorem yields the following results:

**Theorem 3** (Inverse function theorem). *Let $g$ be a continuously differentiable function from $\Omega \subseteq \mathbb{R}$ to $\mathbb{R}$. Fix a $\hat{y} \in \Omega$. If $g'(\hat{y}) \neq 0$, there exists a neighborhood $H$ of $\hat{y}$, a neighborhood $G$ of $g(\hat{y})$, and function $f$ defined on $G$ (the inverse of $g$), such that*

$$g(f(x)) = x, \quad \forall x \in G. \tag{8}$$

*Moreover,*

$$f'(x) = \frac{1}{g'(f(x))}, \quad \forall x \in G. \tag{9}$$

**Proof of Theorem 3:** Let $g$ be as in the theorem, and consider the map $F(x, y) = x - g(y)$. Set $\hat{x} = g(\hat{y})$. Then $F_y(\hat{x}, \hat{y}) = g'(\hat{y}) \neq 0$. Theorem 2 asserts the existence of a function $f$ such that $F(x, f(x)) = \hat{x} - g(\hat{y}) = 0$ for all $x$ in some neighborhood $G$ of $\hat{x} = g(\hat{y})$. In other words, $0 = x - g(f(x))$ for all $x \in G$, which is (8). To obtain (9), simply differentiate (8). $\square$

**Example:** Consider the function $F(x, y) = x - y^2$ and the level set $\Gamma = \{(x, y) : F(x, y) = 0\}$. Then $F_y(x, y) = -2y$ so $F_y(x, y) \neq 0$ as long as $y \neq 0$. In other words, the parabola $x = y^2$ can locally be parameterized as a function of $x$ at every point $x > 0$, but not at $x = 0$. Similarly, the function $y = \sqrt{x}$ be be locally inverted for every $x > 0$, but not in any neighborhood of 0.

**Example:** Consider the function $F(x, y) = x^2 - y^2$ and the level set $\Gamma = \{(x, y) : F(x, y) = 0\}$. What happens at the origin?

**Example:** Consider the function $g(y) = y^3$. We have $g'(0) = 0$, so Theorem 3 cannot assure that $g$ is locally invertible at $y = 0$. It is, however, since $f(x) =$

$|x|^{1/3}\,\mathrm{sign}(x)$ is a well-defined global inverse. We see that while the conditions of Theorem 2 and 3 are sufficient, they are certainly not necessary.

**Example:** Consider the function $F(x_1, x_2, y) = x_1^2 + x_2^2 + y^2$ and the level set $A = \{(x_1, x_2, y) : F(x_1, x_2, y) = 1\}$, the unit sphere in $\mathbb{R}^3$. Fix a point $(\hat{x}_1, \hat{x}_2, \hat{y})$ such that $x_1^2 + x_2^2 < 1$. Then $F_y(\hat{x}_1, \hat{x}_2, \hat{y}) = 2\hat{y} \neq 0$ so the implicit function theorem implies that $A$ can be locally parameterized as a function $y = f(x_1, x_2)$ in some neighborhood of $(\hat{x}_1, \hat{x}_2)$. The formula (7) says that

$$(f_1,\, f_2) = -\frac{1}{F_y}(F_1, F_2) = -\frac{1}{2y}(2x_1, 2x_2). \tag{10}$$

Note that equation (10) enables the evaluation of $\nabla f(\hat{x}_1, \hat{x}_2)$ without explicitly constructing $f$.

## 6. General Implicit Function Theorem

**Theorem 4** (Implicit Function Theorem)**.** *Let $X$, $Y$, and $Z$ be Banach spaces and let $\Omega$ be an open subset of $X \times Y$. Let $F$ be a continuously differentiable map from $\Omega$ to $Z$. If $(\hat{x}, \hat{y}) \in \Omega$ is a point such that $D_y F(\hat{x}, \hat{y})$ is a bounded, invertible, linear map from $Y$ to $Z$, then there is an open neighborhood $G$ of $\hat{x}$, and a unique function $f : G \to Y$ such that*

$$F(x, f(x)) = F(\hat{x}, \hat{y}), \qquad \forall\, x \in G.$$

*Moreover, $f$ is continuously differentiable, and*

$$f'(x) = -[F_y(x, f(x))]^{-1}\, F_x(x, f(x)).$$

By applying the Implicit Function Theorem to the function $F(x, y) = x - g(y)$, we immediately obtain the Inverse Function Theorem:

**Theorem 5** (Inverse Function Theorem)**.** *Suppose that $X$ and $Y$ are Banach spaces, and that $\Omega$ is an open subset of $Y$. Let $g$ be a continuously differentiable function from $\Omega$ to $X$. Fix $\hat{y} \in \Omega$. If $g'(\hat{y})$ has a bounded inverse, then there exists a neighborhood $G$ of $g(\hat{y})$, and a unique function $f$ from $G$ to $\Omega$ such that*

$$g(f(x)) = x, \qquad \forall\, x \in G.$$

*Moreover, $g$ is continuously differentiable, and*

$$f'(x) = [g'(f(x))]^{-1}, \qquad \forall\, x \in G.$$

**Example (13.21 from the text book):** Consider the ODE

$$\ddot{u} + \sin u = h. \tag{11}$$

We assume that $h$ is a periodic function with period $T$, and seek a solution $u$ that also has period $T$. We cast (11) as a functional equation by introducing the function spaces

$$X = \{u \in C^2(\mathbb{R}) : \; u(t) = u(t+T)\ \forall t \in \mathbb{R}\},$$
$$Y = \{u \in C(\mathbb{R}) : \; u(t) = u(t+T)\ \forall t \in \mathbb{R}\},$$

and the non-linear map

$$f:\ X\rightarrow Y:\ u\mapsto \ddot{u}+\sin\,u.$$

Then (11) can be formulated as follows: Given $h\in Y$, determine $u\in X$ such that

$$f(u)=h. \tag{12}$$

For $h=0$, equation (12) clearly has the solution $u=0$. Moreover, $f$ is continuously differentiable in some neighborhood of 0 since

$$f(u+v)=\ddot{u}+\ddot{v}+\sin(u+v)=\underbrace{\ddot{u}+\sin\,u}_{=f(u)}+\underbrace{\ddot{v}+(\cos\,u)\,v}_{=(f'(u))\,v}+O(||v||_Y^2).$$

The map $f'(0)\in\mathcal{B}(X,Y)$ has a continuous inverse if and only if the equation

$$\ddot{v}+v=h$$

has a unique solution for every $h\in Y$. We know from basic ODE theory that this is true if and only if $T\neq 2\pi n$ for any integer $n$. The inverse function theorem then states the following: For every $T$ that is not an integer multiple of $2\pi$, there exists an $\varepsilon>0$ such that (11) has a unique, $C^2$, $T$-periodic solution for every continuous, $T$-periodic function $h$ such that $||h||_{\rm u}<\varepsilon$.

**Example:** Set $I=[0,1]$, and consider the map

$$f:\ L^2(I)\rightarrow L^1(I):\ u\mapsto (1/2)\,u^2.$$

Then $f$ is continuously differentiable, with

$$f'(u):\ L^2(I)\rightarrow L^1(I):\ \varphi\mapsto u\,\varphi.$$

Note that

$$||f'(u)\,\varphi||_1=\int|u\,\varphi|\leq||u||_2\,||\varphi||_2,$$

so $f'(u)$ is continuous. Set $\hat{u}=1$, and $\hat{v}=f(\hat{u})=1$. Then

$$f'(\hat{u}):\ \varphi\mapsto\varphi,$$

which certainly appears to be invertible. However, the map $f$ cannot be invertible in any neighborhood of $\hat{u}$. To see this, we note that for any $\varepsilon>0$, the functions

$$u_1=2\chi_{[0,\delta]}+\chi_{[\delta,1]},\quad\text{and}\quad u_2=-2\chi_{[0,\delta]}+\chi_{[\delta,1]},$$

satisfy $||u_j-\hat{u}||_2<\varepsilon$ provided that $\delta<\varepsilon^2/9$. Consequently, $f$ cannot be injective on any neighborhood of $\hat{u}$. There is no contradiction to the inverse function theorem, however, since $f'(x):\ X\rightarrow Y$, is not continuously invertible. There are several (essentially equivalent) ways to verify this. The easiest is to note that $f'(x)$ is not onto, since $L^2(I)$ is a strict subset of $L^1(I)$ (example: $x^{-1/2}\in L^1\backslash L^2$). Alternatively, one could show that the map $\varphi\mapsto\varphi$ is not a continuous map from $L^1$ to $L^2$. For instance, set $\varphi_n=n\chi_{[0,1/n]}$. Then $||\varphi_n||_1=1$, but $||\varphi_n||_2=\sqrt{n}\rightarrow\infty$.