Applied Analysis (APPM 5450): Midterm 1
5.00pm – 6.20pm, Feb. 20, 2006. Closed books.

Problem 1: No motivation is required.

(a) Set $H = L^2(-\pi, \pi)$, and define $[Au](x) = (1 + x^3)\, u(x)$. What is $\sigma(A)$? (2p)

(b) Consider the Hilbert space $H = l^2(\mathbb{N})$. Give an example of an operator $A \in \mathcal{B}(H)$ that is both unitary and skew-symmetric. (2p)

(c) Let H be a Hilbert space and let $A \in \mathcal{B}(H)$. Give a formula that relates the range of A to the kernel of A^*. (2p)

(d) Let H be a Hilbert space. Is it true that any set that is bounded and closed in the norm topology is necessarily compact in the weak topology? (2p)

Problem 2: Consider the Hilbert space $H = L^2(I)$, where $I = [-\pi,\, \pi]$. Let $k(x, y)$ be a continuous function on $I \times I$, and let the operator $A \in \mathcal{B}(H)$ be defined by

$$[Au](x) = \int_{-\pi}^{\pi} k(x, y)\, u(y)\, dy.$$

(a) Construct the operator A^*. (3p)

(b) Give a necessary and sufficient condition for A to be self-adjoint. (2p)

Problem 3: Consider the Hilbert space $H = L^2(I)$, where $I = [-\pi, \pi]$. Please motivate your answers to the following questions:

(a) Does the sequence $\varphi_n(x) = x\, \sin(nx)$ converge weakly in H? (3p)

(b) Does the sequence $\psi_n(x) = n\, \sin(nx)$ converge weakly in H? (3p)

Problem 4: Consider the Hilbert space $H = l^2(\mathbb{N})$. Let $L \in \mathcal{B}(H)$ denote the left-shift operator, defined by

$$L\,(x_1,\, x_2,\, x_3, \dots) = (x_2,\, x_3,\, x_4, \dots).$$

Recall that $L^* = R$, where R is the right-shift operator,

$$R\,(x_1,\, x_2,\, x_3, \dots) = (0,\, x_1,\, x_2, \dots).$$

(a) Set $X = \{z \in \mathbb{C} :\ |z| > 1\}$. Prove that $X \subseteq \rho(L)$, and that $X \subseteq \rho(R)$. (2p)

(b) Set $Y = \{z \in \mathbb{C} :\ |z| < 1\}$. Prove that $Y \subseteq \sigma_{\rm p}(L)$. (2p)

(c) Prove that for any operator A on a Hilbert space H, it is the case that if $\lambda \in \sigma_{\rm r}(A)$, then $\bar{\lambda} \in \sigma_{\rm p}(A^*)$. (3p)

(d) Set $Z = \{z \in \mathbb{C} :\ |z| \leq 1\}$. Prove that $\sigma(L) = Z$. (2p)

The last problem is deliberately given only a measly amount of points. It would probably not be a good idea to try it unless you finish the first four rapidly.

Problem 5: Let A be an $n \times n$ Hermitian matrix and set $B = (A + i\,I)^{-1}\,(A - i\,I)$, where $i = \sqrt{-1}$. Prove that $||B\,x|| = ||x||$, where $||x|| = (\sum_{j=1}^{n} |x_j|^2)^{1/2}$. (2p)