Applied Analysis (APPM 5450): Midterm 3
5.00pm – 6.20pm, Apr 24, 2006. Closed books.

Note: In your solutions, explicitly state if you use an integral sign that does **not** refer to a Lebesgue integral. (All integrals on this page are Lebesgue integrals.)

Problem 1: In this question, $(X, \mathcal{A}, \mu)$ denotes a measure space.

(a) What axioms must $\mathcal{A}$ satisfy? (2p)

(b) What axioms must μ satisfy? (2p)

(c) Prove that if $\Omega_1,\, \Omega_2 \in \mathcal{A}$, then $\mu(\Omega_1 \cup \Omega_2) \leq \mu(\Omega_1) + \mu(\Omega_2)$. Given an exact condition for when equality occurs. ("Equality occurs if and only if ...") (2p)

(d) Define the Lebesgue integral of a measurable non-negative function on X. (2p)

(e) Define the "essential supremum" of a measurable function f on X. (2p)

(f) For which extended real numbers p are the simple functions dense in $L^p(\mathbb{R}^d)$? When is $C_c^\infty(\mathbb{R}^d)$ dense in $L^p(\mathbb{R}^d)$? (2p)

Problem 2: Define the real-valued function f on $\mathbb{R}^2$ by $f(x_1, x_2) = x_1\, x_2^2$. Define a tempered distribution T_f by $\langle T_f,\, \varphi\rangle = \int_{\mathbb{R}^2} f(x)\, \varphi(x)\, dx$. What is the Fourier transform of T_f? Motivate your answer carefully. (4p)

Problem 3: Calculate the limit

$$\lim_{n\to\infty} \int_n^{n+1} \sqrt{x}\, \tan(1/\sqrt{x})\, dx.$$

Motivate your answer carefully. (4p)

Problem 4: Recall that for $s \in [0, \infty)$, the Sobolev space $H^s(\mathbb{R}^d)$ is defined as the set of all functions $f \in L^2(\mathbb{R}^d)$ such that $(1 + |t|^2)^{s/2}\, \hat{f}(t) \in L^2$. Prove that if s is large enough, then $H^s(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$. (4p)

Problem: The following problems are worth 2p each.

(a) State the definition of a σ-algebra.

(b) State the definition of a measure.

(c) Let $(X, \mathcal{A}, \mu)$ denote a measure space. Suppose that Ω_1, $\Omega_2 \in \mathcal{A}$. Prove that $\mu(\Omega_1 \cup \Omega_2) \leq \mu(\Omega_1) + \mu(\Omega_2)$. Give a condition for when equality occurs.

Problem: Given a measure space $(X, \mathcal{A}, \mu)$, define (a) the Lebesgue integral of a simple function, and (b), the Lebesgue integral of a measurable non-negative function. (4p)

Problem: Assume that $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in $\mathcal{S}(\mathbb{R})$. Set $\chi_n(x) = \varphi_n(x)\, \psi_n(x)$. Prove that $\chi_n \to \varphi\, \psi$ in $\mathcal{S}$.