Applied Analysis (APPM 5450): Final – Solutions

Problem 1: No motivation required for these questions. 2p each.

(a) State Hölder's inequality.

(b) Define what it means for a sequence $(\varphi_n)_{n=1}^{\infty}$ of Schwartz functions to converge in $\mathcal{S}(\mathbb{R})$.

(c) Let H be a Hilbert space, and let $(A_n)_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{B}(H)$. Define what it means for A_n to converge *strongly* to some operator $A \in \mathcal{B}(H)$.

(d) Let (X, μ) be a σ-finite measure space. For which numbers p in the interval $[1, \infty]$ is it necessarily the case that $(L^p(X, \mu))^* = L^q(X, \mu)$, where q is such that $(1/p) + (1/q) = 1$. For which numbers p is $L^p(X, \mu)$ necessarily reflexive?

(e) Let H be a Hilbert space, and let A be a linear bounded operator on H. Give a formula that relates the range of A to the kernel of A^*.

(f) Let H be a Hilbert space and let $A \in \mathcal{B}(H)$ be a self-adjoint operator. Let H_1 be an invariant subspace of A. Is $H_1^{\perp}$ necessarily an invariant subspace of A? Is $H_1^{\perp}$ necessarily an invariant subspace of A if A is skew-adjoint instead of self-adjoint?

(g) Let H be a Hilbert space, and let $A \in \mathcal{B}(H)$ be self-adjoint and compact. What can you say about $\sigma_c(A)$?

(a) Let $p, q \in [1, \infty]$ be such that $(1/p) + (1/q) = 1$, let (X, μ) be a measure space, let $f \in L^p(X, \mu)$, and let $g \in L^q(X, \mu)$. Then $fg \in L^1(X, \mu)$ and $||f\,g||_1 \le ||f||_p\,||g||_q$.

(b) $\varphi_n \to \varphi$ if for every $\alpha \in \mathbb{N}^d$ and $k \in \mathbb{N}$, we have $\lim_{n\to\infty} ||\varphi - \varphi_n||_{\alpha,k} = 0$, where $||\varphi||_{\alpha,k} = \sup_x (1 + |x|^2)^{k/2} |\partial^\alpha \varphi(x)|$.

(c) $A_n \to A$ strongly if for every $x \in H$ we have $\lim_{n\to\infty} ||A\,x - A_n\,x|| = 0$.

(d) If $p \in [1, \infty)$ then $(L^p)^*$ necessarily equals L^q. If $p \in (1, \infty)$, then L^p is necessarily reflexive.

(e) $\overline{\mathrm{ran}(A)} = (\ker(A^*))^{\perp}$

(f) Yes, and yes.

(g) Either $\sigma_c(A) = \{0\}$ or $\sigma_c(A) = \emptyset$.

Problem 2: Let H be a Hilbert space, and let $P \in \mathcal{B}(H)$ be an operator such that $P^2 = P$. Prove that the statements (S1) and (S2) given below are equivalent: (4p)

(S1): $(\text{ran}(P))^{\perp} = \ker(P)$.

(S2): $\langle P\,x,\, y\rangle = \langle x,\, P\,y\rangle$ for all $x, y \in H$.

(S1) $\Rightarrow$ (S2): Assume that (S1) holds. First note that for any $z \in H$, we have $(I - P)z \in \ker(P)$ since $P(I - P)z = Pz - P^2 z = Pz - Pz = 0$. Then, for $x, y \in H$

$$\langle Px,\, y\rangle = \langle Px,\, Py + (I-P)y\rangle = \langle Px,\, Py\rangle + \langle Px,\, (I-P)y\rangle = \langle Px,\, Py\rangle.$$

The last equality used that $Px \in \text{ran}(P)$, that $(I - P)y \in \ker(P)$ and assumption (S1). Analogously

$$\langle x,\, Py\rangle = \cdots = \langle Px,\, Py\rangle,$$

and so $\langle Px,\, y\rangle = \langle x,\, Py\rangle$.

(S2) $\Rightarrow$ (S1): Assume that (S2) is true. Then

$$x \in \ker(P) \;\Leftrightarrow\; \langle Px, y\rangle = 0\ \forall y \;\Leftrightarrow\; \langle x, Py\rangle = 0\ \forall y \;\Leftrightarrow\; x \in (\text{ran}(P))^{\perp}.$$

Problem 3: Let $\delta \in \mathcal{S}(\mathbb{R})^*$ denote the Dirac delta-function as usual, let δ' denote the distributional derivative of δ, and define for a positive integer n the distribution $T_n \in \mathcal{S}(\mathbb{R})^*$ by $T_n(x) = \sin(n\,x)\,\delta'(x)$.

(a) Calculate the Fourier transform $\hat{T}_n$ of T_n. (2p)

(b) Does the sequence $(\hat{T}_n)_{n=1}^{\infty}$ converge in $\mathcal{S}(\mathbb{R})^*$? (2p)

Hint: You may want to start by simplifying the expression for T_n.

First we simplify the expression for T_n. If $\varphi \in \mathcal{S}$, then

$$\begin{aligned}\langle T_n,\,\varphi\rangle &= \langle \sin(nx)\delta',\,\varphi\rangle = \langle \delta',\,\sin(nx)\varphi\rangle = -\langle \delta,\,\partial(\sin(nx)\varphi)\rangle \\ &- \langle \delta,\,n\cos(nx)\varphi + \sin(nx)\varphi'\rangle = -n\cos(0)\varphi(0) - \sin(n)\varphi'(0) = -n\varphi(0).\end{aligned}$$

Consequently, $T_n = -n\,\delta$.

(a) Since $\hat{\delta} = 1/\sqrt{2\pi}$, we find that $\hat{T}_n = -n/\sqrt{2\pi}$.

(b) If $\varphi \in \mathcal{S}$, then

$$\langle \hat{T}_n,\,\varphi\rangle = \langle -\frac{n}{\sqrt{2\pi}},\,\varphi\rangle = -\frac{n}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x)\,dx.$$

When $\int \varphi \neq 0$, we have $\langle \hat{T}_n,\,\varphi\rangle \to \pm\infty$, so $\hat{T}_n$ cannot converge.

Problem 4: Let $p \in [1, \infty)$, let g be a function in $L^p(\mathbb{R})$, and let $(f_n)_{n=1}^{\infty}$ be measurable functions from $\mathbb{R}$ to $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} |f_n(x)| \leq g(x), \qquad \text{a.e.}$$

Set $h_N = \sum_{n=1}^{N} f_n$. Prove that the sequence $(h_N)_{N=1}^{\infty}$ converges in $L^p(\mathbb{R})$. (5p)

Set $\Omega_1 = \{x : \ |g(x)| < \infty\}$. Then $\mu(\Omega_1^{\rm c}) = 0$, since $g \in L^p$.

Set $\Omega_2 = \{x : \ \sum |f_n(x)| < |g(x)|\}$, then $\mu(\Omega_2^{\rm c}) = 0$ by assumption.

Set $\Omega = \Omega_1 \cap \Omega_2$. Then $\mu(\Omega^{\rm c}) \leq \mu(\Omega_1^{\rm c}) + \mu(\Omega_2^{\rm c}) = 0$.

For $x \in \Omega$, we have $\sum_{n=1}^{\infty} |f_n(x)| < \infty$, so we the following formula defines a finite valued function:

$$h(x) = \begin{cases} \sum_{n=1}^{\infty} f_n(x), & x \in \Omega, \\ 0, & x \in \Omega^{\rm c}. \end{cases}$$

It follows immediately that $|h(x)| \leq g(x)$ for all x, and so $h \in L^p$.

We will prove that $||h - h_N||_p \to 0$ as $N \to \infty$. We have

$$||h - h_N||_p^p = \int_{\mathbb{R}} \Big| h(x) - \sum_{n=1}^{N} f_n(x) \Big|^p \, dx.$$

Note that

(a) $\displaystyle \Big| h(x) - \sum_{n=1}^{N} f_n(x) \Big|^p \to 0, \qquad$ pointwise,

and that

(b) $\displaystyle \Big| h(x) - \sum_{n=1}^{N} f_n(x) \Big|^p \ \leq \ \Big(|h(x)| + \sum_{n=1}^{N} |f_n(x)| \Big)^p \ \leq \ (2\, g(x))^p \ = \ 2^p\, g(x)^p$. Since $g \in L^p$, we know that $\int 2^p\, g^p < \infty$, and so in light of (a) and (b), we can invoke the Lebesgue dominated convergence theorem:

$$\lim_{N \to \infty} ||h - h_N||_p^p = \int_{\mathbb{R}} \lim_{N \to \infty} \Big| h(x) - \sum_{n=1}^{N} f_n(x) \Big|^p \, dx = \int_{-\infty}^{\infty} 0 \, dx = 0.$$

Problem 5: Consider the Hilbert space $H = L^2(\mathbb{T})$, let $a \in (0, \pi)$ be a real number, and define the operator $T \in \mathcal{B}(H)$ by

$$[T\,u](x) = \frac{1}{2}\big(u(x-a) + u(a-x)\big).$$

(a) Construct T^* and indicate whether T is self-adjoint. (2p)

(b) Prove that T is not unitary. Is T normal? (2p)

(c) Specify infinite dimensional subspaces H_1 and H_2 of H such that the map $T : H_1 \to H_2$ is a unitary operator. (2p)

(d) Let $\mathcal{F} : H \to l^2(\mathbb{Z})$ denote the Fourier transform. Determine the operator $\hat{T} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ given by $\hat{T} = \mathcal{F}\,T\,\mathcal{F}^{-1}$. (2p)

(e) Determine $\sigma(T)$. As far as you can, classify the different parts of the spectrum as belonging to the point, continuous, or residual spectrum. (3p)

First note that $T = S_a\,P$, where

$$[Pu](x) = \frac{1}{2}(u(x) + u(-x)), \qquad \text{and} \qquad [S_a u](x) = u(x-a).$$

(In other words, P is the orthogonal projection onto the even functions, and S_a is a simple shift operator.)

(a) $T^* = (S_a\,P)^* = P^*\,S_a^* = P\,S_{-a}$. In other words,

$$[T^* u](x) = \frac{1}{2}\big(u(x+a) + u(a-x)\big).$$

We see that T is not self-adjoint.

(b) $T^*\,T = P\,S_{-a}\,S_a\,P = P\,P = P$ and $T\,T^* = S_a\,P\,P\,S_{-a} = S_a\,P\,S_{-a}$ so T is neither unitary nor normal. (Note that $[S_a\,P\,S_{-a}\,u](x) = \frac{1}{2}u(x) + \frac{1}{2}u(-x+2a)$.)

(c) Let H_1 denote the subspace of even functions, and let H_2 denote the space of functions that are even around the point $x = a$ (so that $f \in H_2 \;\Leftrightarrow\; f(a-x) = f(a+x)$ for all x). Then $T|_{H_1} = S_a$ which is clearly unitary.

(d) Let α_n denote the Fourier coefficients of a function u, and set $v = T\,u$. Then we calculate Fourier coefficients γ_n of v:

$$\begin{aligned}\gamma_n &= \beta \int_{\mathbb{T}} e^{-inx}\frac{1}{2}\big(u(x-a) + u(a-x)\big)\,dx \\ &= \beta \int_{\mathbb{T}} e^{-in(y+a)}\frac{1}{2}u(y)\,dy + \beta \int_{\mathbb{T}} e^{-in(a-y)}\frac{1}{2}u(y)\,dy = e^{-ina}\frac{1}{2}(\alpha_n + \alpha_{-n}).\end{aligned}$$

So $\hat{T} :\ (\alpha_n) \mapsto (\gamma_n)$ where $\gamma_n = e^{-ina}(\alpha_n + \alpha_{-n})/2$.

(e) Since $\mathcal{F}$ is a unitary map, the spectrum of T is identical to the spectrum of $\hat{T}$. We can therefore answer the question by determining the spectrum of $\hat{T}$.

Recall that a number $\lambda \in \mathbb{C}$ belongs to $\sigma(\hat{T})$ if the operator $\hat{T} - \lambda I$ does not have a bounded inverse. We therefore consider the equation

$$(1) \qquad \big(\hat{T} - \lambda I\big)\, \alpha = \gamma.$$

Setting $\mu = e^{-ina}$, we write equation (1) componentwise as

$$(2) \qquad (1 - \lambda)\, \alpha_0 = \gamma_0,$$

$$(3) \qquad \begin{bmatrix} \frac{1}{2}\mu - \lambda & \frac{1}{2}\mu \\ \frac{1}{2}\bar{\mu} & \frac{1}{2}\bar{\mu} - \lambda \end{bmatrix} \begin{bmatrix} \alpha_n \\ \alpha_{-n} \end{bmatrix} = \begin{bmatrix} \gamma_n \\ \gamma_{-n} \end{bmatrix}, \qquad n \neq 0.$$

Case 1 – $\lambda = 1$: In this case, equation (2) does not have a solution. In fact, if v is any constant vector, then $T\, v = v$, so $1 \in \sigma_{\rm p}(T)$.

Case 2 – $\lambda = 0$: In this case, equation (3) is singular. In fact, if v is an odd function (so that $\alpha_n = -\alpha_{-n}$) then $T\, v = 0$, so $0 \in \sigma_{\rm p}(T)$.

Case 3 – $\lambda \neq 0, 1$: For this case, equation (2) is invertible, and (3) is invertible if and only if

$$0 \neq \big(\tfrac{1}{2}\mu - \lambda\big)\big(\tfrac{1}{2}\bar{\mu} - \lambda\big) - \tfrac{1}{4}\mu\bar{\mu}.$$

Simplifying, we obtain the equation

$$0 \neq \lambda\, \big(\lambda - \tfrac{1}{2}(\mu + \bar{\mu})\big).$$

We find that (3) is singular if $\lambda = 0$ or if

$$\lambda = \tfrac{1}{2}\big(\mu + \bar{\mu}\big) = \cos(na).$$

The eigenvector corresponding to $\lambda = \cos(na)$ is

$$v_n = \alpha_n\, e^{inx} + \alpha_{-n}\, e^{-inx} = \mu e^{inx} + \bar{\mu} e^{-inx} = e^{in(x-a)} + e^{-in(x-a)} = 2\cos(nx - na).$$

Thus $\sigma_{\rm p}(T) = \{0\} \cup \{1\} \cup \{\cos(na)\}_{n=1}^{\infty}$.

Remark: If you got this far, you got full credit.

If $\lambda \in \mathbb{C}$ is a number such that $\text{dist}(\lambda, \sigma_{\rm p}(T)) > 0$, then the system (2,3) is boundedly invertible, so $\lambda \in \rho(T)$. In contrast, if $\lambda \in \overline{\sigma_{\rm p}(T)}$ then $\hat{T} - \lambda I$ is injective, but not boundedly invertible. In fact, if $\lambda_{n_j} \to \lambda$ as $j \to \infty$, we have $||(T - \lambda I)\, v_{n_j}|| = ||(\lambda_{n_j} - \lambda)\, v_{n_j}|| \to 0$ so $\lambda \in \sigma_{\rm c}(T)$.

To summarize:

$\sigma_{\rm p}(T) = \{0\} \cup \{1\} \cup \{\cos(na)\}_{n=1}^{\infty}$

$\sigma_{\rm c}(T) = \overline{\sigma_{\rm p}(T)} \setminus \sigma_{\rm p}(T)$

$\sigma_{\rm r}(T) = \emptyset$

Remark 1: If a/π is a rational number, then $\sigma_{\rm p}(T)$ is finite, and $\sigma(T) = \sigma_{\rm p}(T)$.

Remark 2: Since T is not normal, its eigenvalue decomposition is not of much value. Of more interest is the decomposition $T = S_a\, P$. It is an analogue of the singular value decomposition of T and specifies exactly the action of T, its null-space, its range, and so on.