

Homework set 14 — APPM5450, Spring 2007

From the book: 12.8, 12.16, 12.17, 12.18. Optional: 12.13, 12.14, 12.15.

Problem 1: Let λ be a real number such that $\lambda \in (0, 1)$, and let a and b be two non-negative real numbers. Prove that

$$(1) \quad a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda) b,$$

with equality iff $a = b$.

Hint: Consider the case $b = 0$ first. When $b \neq 0$, change variables to $t = a/b$.

Problem 2: [Hölder's inequality] Suppose that p is a real number such that $1 < p < \infty$, and let q be such that $p^{-1} + q^{-1} = 1$. Let (X, μ) be a measure space, and suppose that $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$. Prove that $fg \in L^1(X, \mu)$, and that

$$(2) \quad \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Prove that equality holds iff $\alpha|f|^p = \beta|g|^q$ for some α, β such that $\alpha\beta \neq 1$.

Hint: Consider first the case where $\|f\|_p = 0$ or $\|g\|_q = 0$. For the case $\|f\|_p \|g\|_q \neq 0$, use (1) with

$$a = \left| \frac{f(x)}{\|f\|_p} \right|^p, \quad b = \left| \frac{g(x)}{\|g\|_q} \right|^q, \quad \lambda = \frac{1}{p}.$$

Problem 3: [Minkowski's inequality] Let (X, μ) be a measure space, and let p be a real number such that $1 \leq p \leq \infty$. Prove that for $f, g \in L^p(X, \mu)$,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Hint: Consider the cases $p = 1, \infty$ separately. For $p \in (1, \infty)$, note that

$$(3) \quad |f(x) + g(x)|^p \leq (|f(x)| + |g(x)|) |f(x) + g(x)|^{p-1}, \quad \forall x \in X.$$

Then integrate both sides of (3) and apply (2) to the right hand side.