Homework set 8 — APPM5450, Spring 2007 — Hints

11.5: Note that

$$\frac{1}{x+i\varepsilon} = \frac{x}{\varepsilon^2+x^2} - i\frac{\varepsilon}{\varepsilon^2+x^2}.$$

Fix a $\varphi \in \mathcal{S}$. You need to prove that

$$\lim_{\varepsilon\to 0}\langle i\frac{\varepsilon}{\varepsilon^2+x^2},\, \varphi\rangle \to -i\pi\varphi(0). \tag{1}$$

and that

$$\lim_{\varepsilon\to 0}\langle \frac{x}{\varepsilon^2+x^2},\, \varphi\rangle \to \langle \mathrm{PV}\left(\frac{1}{x}\right),\, \varphi\rangle, \tag{2}$$

Proving (1) is simple:

$$\langle i\frac{\varepsilon}{\varepsilon^2+x^2},\, \varphi\rangle = \int_{-\infty}^{\infty} i\frac{\varepsilon}{\varepsilon^2+x^2}\,\varphi(x)\,dx = \{\text{Set } x=\varepsilon y\} = \cdots$$

For (2) we need to work a bit more (unless I overlook a simpler solution)

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\langle \frac{x}{\varepsilon^2+x^2},\, \varphi\rangle - \langle \mathrm{PV}\left(\frac{1}{x}\right),\, \varphi\rangle \\
&\qquad = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{x}{\varepsilon^2+x^2}\,\varphi(x)\,dx - \lim_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\frac{1}{x}\,\varphi(x)\,dx \\
&= \underbrace{\lim_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\left(\frac{x}{\varepsilon^2+x^2} - \frac{1}{x}\right)\varphi(x)\,dx}_{=S_1} + \underbrace{\lim_{\varepsilon\to 0}\int_{|x|\leq\sqrt{\varepsilon}}\frac{x}{\varepsilon^2+x^2}\,\varphi(x)\,dx}_{=S_2}.
\end{aligned}$$

First we bound $|S_1|$. Note that when $|x|\geq\sqrt{\varepsilon}$, we have

$$\left|\frac{x}{\varepsilon^2+x^2} - \frac{1}{x}\right| = \frac{\varepsilon^2}{|x|(\varepsilon^2+x^2)} \leq \frac{\varepsilon^2}{|x|^3} \leq \frac{\varepsilon^2}{\varepsilon^{3/2}} = \sqrt{\varepsilon}.$$

Consequently,

$$\begin{aligned}
|S_1| &\leq \limsup_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\left|\frac{x}{\varepsilon^2+x^2} - \frac{1}{x}\right|\,|\varphi(x)|\,dx \\
&\qquad \leq \limsup_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\sqrt{\varepsilon}\frac{1}{(1+|x|^2)}\underbrace{|(1+|x|^2)\varphi(x)|}_{\leq||\varphi||_{0,2}}\,dx = 0.
\end{aligned}$$

In bounding S_2 we use that

$$\int_{|x|\leq\sqrt{\varepsilon}}\frac{x}{\varepsilon^2+x^2}\,\varphi(0)\,dx = 0,$$

and that

$$|\varphi(x)-\varphi(0)| \leq |x|\,||\varphi'||_{\mathrm{u}} \leq |x|||\varphi||_{1,0},$$

to obtain

$$
\begin{aligned}
|S_2| = \left|\lim_{\varepsilon\to 0}\int_{|x|\leq\sqrt{\varepsilon}} \frac{x}{\varepsilon^2+x^2}\,(\varphi(x)-\varphi(0))\,dx\right| \\
\leq \limsup_{\varepsilon\to 0}\int_{|x|\leq\sqrt{\varepsilon}} \underbrace{\frac{|x|}{\varepsilon^2+x^2}\,|x|}_{\leq 1}\,||\varphi||_{1,0}\,dx = 0.
\end{aligned}
$$

Problem 11.6: We find that

$$
\begin{aligned}
\langle D(\log|x|)\,\varphi\rangle &= -\langle \log|x|\,\varphi'\rangle = -\int_{\mathbb{R}} \log|x|\,\varphi'(x)\,dx \\
&= -\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\varepsilon} \log(-x)\varphi'(x)\,dx + \int_{\varepsilon}^{\infty} \log(x)\varphi'(x)\,dx\right].
\end{aligned}
$$

Now simply perform partial integration in each term separately.

Problem 11.7: First prove that $x\cdot\delta(x) = 0$ and that $x\cdot \mathrm{PV}(1/x) = 1$ (using the regular rules for the product between a polynomial and a Schwartz function). Suppose that $\cdot$ is distributive and can pair any two distributions. Then on the one hand we would have

$$\delta(x)\cdot x\cdot \mathrm{PV}(1/x) = \delta(x)\cdot(x\cdot\mathrm{PV}(1/x)) = \delta(x)\cdot 1 = \delta(x).$$

But we would also have

$$\delta(x)\cdot x\cdot \mathrm{PV}(1/x) = (x\cdot\delta(x))\cdot\mathrm{PV}(1/x) = 0\cdot\mathrm{PV}(1/x) = 0.$$

This is a contradiction.

Problem 11.8: Fix $\varphi\in\mathcal{S}$. Set $\alpha = \int\varphi$, and define

$$\psi(x) = \int_{-\infty}^{x} \big(\varphi(z) - \alpha\,\omega(z)\big)\,dz. \tag{3}$$

Obviously, $\psi\in C^\infty$, and

$$\varphi(x) = \alpha\omega(x) + \psi'(x). \tag{4}$$

Moreover, we find that if $n\geq 1$, then

$$
\begin{aligned}
||\psi||_{n,k} &= ||(1+|x|^2)^{k/2}\psi^{(n)}||_{\mathrm{u}} \\
&= ||(1+|x|^2)^{k/2}(\varphi^{(n-1)} - \alpha\omega^{(n-1)})||_{\mathrm{u}} \leq ||\varphi||_{n-1,k} + |\alpha|\,||\omega||_{n-1,k}.
\end{aligned}
$$

It remains to prove that for any k,

$$\sup_x (1+|x|^2)^{k/2}|\psi(x)| < \infty.$$

First consider $x \leq 0$. Then for any k, we have

$$\sup_{x\leq 0}(1+|x|^2)^{k/2}|\psi(x)|$$
$$\leq \limsup_{x\leq 0}\Big[(1+|x|^2)^{k/2}\int_{-\infty}^{x}\frac{1}{(1+|y|^{(k+2)/2})}||\varphi||_{0,k+2}\,dy + |\alpha|(1+|x|^2)^{k/2}\int_{-\infty}^{x}\frac{1}{(1+|y|^{(k+2)/2})}||\omega||_{0,k+2}\,dy\Big] < \infty.$$

To prove the corresponding estimate for $x \geq 0$, we use that since

$$\underbrace{\int_{-\infty}^{x}\big(\varphi(z)-\alpha\,\omega(z)\big)\,dz}_{=\psi(x)}+\int_{x}^{\infty}\big(\varphi(z)-\alpha\,\omega(z)\big)\,dz=0,$$

we can also express ψ as

$$\psi(x) = -\int_{x}^{\infty}\big(\varphi(z)-\alpha\,\omega(z)\big)\,dz.$$

Then proceed as in the bound for $x \leq 0$.

Problem 1:

$$\langle D\,f,\varphi\rangle = -\langle f,\varphi'\rangle = -\int_{-\infty}^{0}(-x)\varphi'(x)\,dx - \int_{0}^{\infty}x\varphi'(x)\,dx$$
$$= \underbrace{[x\varphi(x)]_{-\infty}^{0}}_{=0} - \int_{-\infty}^{0}\varphi(x)\,dx - \underbrace{[x\varphi(x)]_{0}^{\infty}}_{=0} + \int_{-\infty}^{0}\varphi(x)\,dx = \langle g,\varphi\rangle,$$

where

$$g(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0. \end{cases}$$

So $D\,f = g$. (Note that the value of $g(0)$ is irrelevant, any finite value can be assigned.) To comute $D^2\,f$, simply differentiate g in the same way. You should find that $D^2\,f = 2\delta$.

Problem 2: This is a fairly straight-forward application of the definitions.

Problem 3: Define for $n = 1, 2, 3, \ldots$, the functions

$$\chi_n(x) = \begin{cases} 1 & x \in \left[n-\frac{1}{4^n},\, n\right], \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$f(x) = \sum_{n=1}^{\infty} 2^n\,\chi_n(x).$$

Now prove that both (2) and (3) hold for any k.