Applied Analysis (APPM 5450): Midterm 3

5.00pm – 6.25pm, April 23, 2007. Closed books.

Problem 1: Pick out the true statements from the list below. One point each, no motivation required.

(a) If $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}$.

(b) If $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}^*$.

(c) If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in C_{\rm b}(\mathbb{R}^d)$.

(d) If $f \in H^s(\mathbb{R}^d)$ and $s > 1/2$, then $f \in C_{\rm b}(\mathbb{R}^d)$.

(e) If $f \in C_0(\mathbb{R}^d)$, then $\hat{f} \in L^2(\mathbb{R}^d)$.

(f) If $f, g \in L^2(\mathbb{R}^d)$, then $\langle f,\, g\rangle_{L^2(\mathbb{R}^d)} = \langle \hat{f},\, \hat{g}\rangle_{L^2(\mathbb{R}^d)}$.

Problem 2: Suppose that $(a_n)_{n=1}^{\infty}$ are real numbers such that $\sum_{n=1}^{\infty} |a_n| < \infty$. Set $f(x) = \sum_{n=1}^{\infty} a_n\, e^{i\,n\,x}$. Is it necessarily the case that $\int_{-\pi}^{\pi} f(x)\,dx = 0$? Motivate your answer. (4p)

Problem 3: For $n = 1, 2, 3, \dots$, set $T_n(x) = \sin(n\,x)\,\chi_{[-n,\,n]}(x)$. Does the sequence $(T_n)_{n=1}^{\infty}$ converge in $\mathbb{S}^*(\mathbb{R})$? Motivate your answer. (4p)

Problem 4: Let f and h be functions in $L^2(\mathbb{R})$. Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of functions in $L^2(\mathbb{R})$ that converges *pointwise* to f. Set

$$\alpha_n = \int_{\mathbb{R}} f_n(x)\,h(x)\,dx, \qquad \text{and} \qquad \alpha = \int_{\mathbb{R}} f(x)\,h(x)\,dx.$$

(a) Give examples of functions f, h, and $(f_n)_{n=1}^{\infty}$ as described above such that the numbers α_n ***do not*** converge to α. (3p)

(b) Suppose that $|f_n(x)| \leq 1/(1 + |x|)$ for all x. Prove that then $\alpha_n \to \alpha$. (3p)

Problem 5 is deliberately given only a small number of points. It's probably only worth attempting if you have time to spare.

Problem 5: Let X be a set and let d be a metric on X. We define a collection $\mathcal{S}$ of subsets of X by saying that $\Omega \in \mathcal{S}$ if and only if for every $x \in \Omega$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \Omega$, where $B_\varepsilon(x) = \{y \in X :\ d(x, y) < \varepsilon\}$.

The following questions are 1p each. Motivate your answers to (b) and (c) briefly.

(a) State the axioms that a σ-algebra must satisfy.

(b) Give an example of an uncountable set X and a metric d such that $\mathcal{S}$ is a σ-algebra.

(c) Give an example of an uncountable set X and a metric d such that $\mathcal{S}$ is not a σ-algebra.