Applied Analysis (APPM 5450) — Midterm 3 — Solutions
5.00pm – 6.25pm, April 23, 2007. Closed books.

Problem 1: Pick out the true statements from the list below. One point each, no motivation required.

(a) If $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}$.

(b) If $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}^*$.

(c) If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in C_\mathrm{b}(\mathbb{R}^d)$.

(d) If $f \in H^s(\mathbb{R}^d)$ and $s > 1/2$, then $f \in C_\mathrm{b}(\mathbb{R}^d)$.

(e) If $f \in C_0(\mathbb{R}^d)$, then $\hat{f} \in L^2(\mathbb{R}^d)$.

(f) If $f, g \in L^2(\mathbb{R}^d)$, then $\langle f,\, g\rangle_{L^2(\mathbb{R}^d)} = \langle \hat{f},\, \hat{g}\rangle_{L^2(\mathbb{R}^d)}$.

(a) True. (Since $\mathcal{F}$ is continuous on $\mathcal{S}$.)

(b) True. (Since $\mathcal{F}$ is continuous on $\mathcal{S}$, we know that $\hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}$; and since convergence in $\mathcal{S}$ implies convergence in $\mathcal{S}^*$, it follows that $\hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}^*$ as well.)

(c) True. (The Riemann-Lebesgue lemma states that in fact $\hat{f} \in C_0(\mathbb{R}^d)$.)

(d) Not true unless $d = 1$. (In the general case, $s > d/2$ is required.)

(e) Not true. (If it were, then we'd have $f \in L^2$ since $\mathcal{F}^{-1}$ is a unitary map on L^2. But not every function in C_0 belongs to L^2.)

(f) True. ($\mathcal{F}$ is a unitary map on $L^2(\mathbb{R}^d)$.)

Problem 2: Suppose that $(a_n)_{n=1}^\infty$ are real numbers such that $\sum_{n=1}^\infty |a_n| < \infty$. Set $f(x) = \sum_{n=1}^\infty a_n\, e^{i\,n\,x}$. Is it necessarily the case that $\int_{-\pi}^{\pi} f(x)\,dx = 0$? Motivate your answer. (4p)

Yes, $\int f = 0$. To prove this, set $f_N(x) = \sum_{n=1}^N a_n\, e^{i\,n\,x}$. Then

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \left(\lim_{N\to\infty} f_N(x)\right)\,dx. \tag{1}$$

Now set

$$g(x) = \sum_{n=1}^\infty |a_n|.$$

Then $|f_N(x)| \leq g(x)$ for all x, and $\int_{-\pi}^{\pi} g(x)\,dx < \infty$. The Lebesgue dominated convergence theorem now allows us to swap the integral and the limit in (1), and so

$$\int_{-\pi}^{\pi} f(x)\,dx = \lim_{N\to\infty} \int_{-\pi}^{\pi} f_N(x)\,dx.$$

Finally note that

$$\int_{-\pi}^{\pi} f_N(x)\,dx = \int_{-\pi}^{\pi} \sum_{n=1}^N a_n\, e^{i\,n\,x}\,dx = \sum_{n=1}^N a_n \int_{-\pi}^{\pi} e^{i\,n\,x}\,dx = 0,$$

since $\displaystyle\int_{-\pi}^{\pi} e^{i\,n\,x}\,dx = 0$ for any positive integer n.

Problem 3: For $n = 1, 2, 3, \ldots$, set $T_n(x) = \sin(n\,x)\,\chi_{[-n,\,n]}(x)$. Does the sequence $(T_n)_{n=1}^{\infty}$ converge in $\mathbb{S}^*(\mathbb{R})$? Motivate your answer. (4p)

Fix $\varphi \in \mathcal{S}$. Then

$$\begin{aligned}
|\langle T_n,\, \varphi\rangle| &= \left| \int_{-n}^{n} \sin(n\,x)\,\varphi(x)\,dx \right| \\
&= \left| \left[-\frac{\cos(n\,x)}{n}\varphi(x) \right]_{-n}^{n} + \int_{-n}^{n} \frac{\cos(n\,x)}{n}\,\varphi'(x)\,dx \right| \\
&= \left| -\frac{\cos(n^2)}{n}\varphi(n) + \frac{\cos(n^2)}{n}\varphi(-n) + \int_{-n}^{n} \frac{\cos(n\,x)}{n}\,\varphi'(x)\,dx \right| \\
&\leq \frac{|\varphi(n)|}{n} + \frac{|\varphi(-n)|}{n} + \frac{1}{n}\int_{-\infty}^{\infty} |\varphi'(x)|\,dx \\
&\leq \frac{2||\varphi||_{0,0}}{n} + \frac{1}{n}\int_{-\infty}^{\infty} \frac{1}{1+x^2}(1+x^2)\,|\varphi'(x)|\,dx \\
&\leq \frac{2||\varphi||_{0,0}}{n} + \frac{1}{n}\pi||\varphi||_{1,2}.
\end{aligned}$$

Consequently, $\langle T_n,\, \varphi\rangle \to 0$ as $n \to \infty$, and so $T_n \to 0$ in $\mathcal{S}^*$.

Problem 4: Let f and h be functions in $L^2(\mathbb{R})$. Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of functions in $L^2(\mathbb{R})$ that converges *pointwise* to f. Set

$$\alpha_n = \int_{\mathbb{R}} f_n(x)\, h(x)\, dx, \qquad \text{and} \qquad \alpha = \int_{\mathbb{R}} f(x)\, h(x)\, dx.$$

(a) Give examples of functions f, h, and $(f_n)_{n=1}^{\infty}$ as described above such that the numbers α_n ***do not*** converge to α. (3p)

(b) Suppose that $|f_n(x)| \leq 1/(1+|x|)$ for all x. Prove that then $\alpha_n \to \alpha$. (3p)

(a) One example is $h(x) = 1/(1+|x|)$ and $f_n(x) = n^2\, \chi_{[n,n+1]}(x)$. Then $f_n \to 0$ pointwise, so $\alpha = 0$, but

$$\alpha_n = \int_n^{n+1} n^2 \frac{1}{1+x}\, dx \geq \frac{n^2}{n+1} \to \infty.$$

(b) Set $u_n(x) = f_n(x)\, h(x)$ and $u(x) = f(x)\, h(x)$. Then $u_n \to u$ pointwise. Setting $g(x) = (1/(1+|x|))\, |h(x)|$, we have $|u_n(x)| \leq g(x)$ for all x. Moreover, a simple application of the Cauchy-Schwartz inequality yields

$$\int_{\mathbb{R}} g(x)\, dx = \int_{\mathbb{R}} \frac{1}{1+|x|}\, |h(x)|\, dx \leq \left[\int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dx \int_{\mathbb{R}} |h(x)|^2\, dx\right]^{1/2},$$

which is finite since both h and $(1+|x|)^{-1}$ are members of $L^2(\mathbb{R})$.[1]

Now according to the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \int_{\mathbb{R}} u_n(x)\, dx = \int_{\mathbb{R}} \left(\lim_{n\to\infty} u_n(x)\right)\, dx \int_{\mathbb{R}} f(x)\, h(x)\, dx = \alpha.$$

[1] Cauchy-Schwartz is a little bit of overkill. The simple inequality $|ab| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$ suffices:

$$\int_{\mathbb{R}} g(x)\, dx = \int_{\mathbb{R}} \frac{1}{1+|x|}\, |h(x)|\, dx \leq \frac{1}{2} \int_{\mathbb{R}} \left(\frac{1}{(1+|x|)^2} + |h(x)|^2\right)\, dx < \infty.$$

This problem has been corrected: The norm that was originally in the problem has been substituted for a metric.

Problem 5: Let X be a set and let d be a metric on X. We define a collection $\mathcal{S}$ of subsets of X by saying that $\Omega \in \mathcal{S}$ if and only if for every $x \in \Omega$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \Omega$, where $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$.

The following questions are 1p each. Motivate your answers to (b) and (c) briefly.

(a) State the axioms that a σ-algebra must satisfy.

(b) Give an example of an uncountable set X and a metric d such that $\mathcal{S}$ is a σ-algebra.

(c) Give an example of an uncountable set X and a metric d such that $\mathcal{S}$ is not a σ-algebra.

(a) See the textbook.

(b) Set $X = \mathbb{R}$ and

$$d(x, y) = \begin{cases} 1, & \text{when } x = y, \\ 0, & \text{when } x \neq y. \end{cases}$$

Then $\mathcal{S}$ is the power set (if Ω is an arbitrary subset, and $x \in \Omega$, then $B_{1/2}(x) \subseteq \Omega$), which trivially implies that it satisfies all the axioms of a σ-algebra.

(c) Set $X = \mathbb{R}$ and $d(x, y) = |x - y|$ (the standard metric on $\mathbb{R}$). Then $\mathcal{S}$ is the standard topology on $\mathbb{R}$, which is not a σ-algebra. To see this, note for instance that $\Omega = (0, \infty) \in \mathcal{S}$, but $\Omega^{\mathrm{c}} = (-\infty, 0] \notin \mathcal{S}$.