Applied Analysis (APPM 5450): Final

7.30 am – 10.00 am, May 6, 2008. Closed books.

Problem 1: In the following questions, the letter H denotes a generic Hilbert space. Motivate your answers briefly. (3p each)

(a) Does the sequence $(\chi_{[n,n+1]})_{n=1}^{\infty}$ converge weakly in $L^3(\mathbb{R})$?

(b) Does there exist a Hilbert space H and an operator $A \in \mathcal{B}(H)$ such that $\sigma(A) = \{1/n\}_{n=1}^{\infty}$?

(c) Let $A \in \mathcal{B}(H)$, and let $\lambda, \mu \in \rho(A)$. Define the resolvent operator by $R_\lambda(A) = (A - \lambda\, I)^{-1}$ as usual and show that $R_\lambda - R_\mu = \alpha\, R_\lambda\, R_\mu$ for some number α. Make sure to specify α.

(d) What can you say about the spectrum of an operator $A \in \mathcal{B}(H)$ that is both self-adjoint and unitary?

(e) Let f_1 and f_2 be functions on $L^2(\mathbb{R})$ such that $||f_1|| = ||f_2|| = 1$ and $\langle f_1,\, f_2\rangle = 0$. Is it possible to say for sure what $||\hat{f}_1 - \hat{f}_2||$ is? If so, give its value.

(f) Which (if any) of the following statements regarding convolutions are always true:
  (i) If $f, g \in \mathcal{S}(\mathbb{R})$, then $f * g \in \mathcal{S}(\mathbb{R})$.
  (ii) If $f, g \in \mathcal{S}^*(\mathbb{R})$, then $f * g \in \mathcal{S}^*(\mathbb{R})$.
  (iii) If $f, g \in L^2(\mathbb{R})$, then $\mathcal{F}[f * g] \in L^1(\mathbb{R})$.

(g) Let $\{\varphi,\, \psi\}$ be a couple of orthonormal vectors in H and define $A \in \mathcal{B}(H)$ via

$$A\, u = \big(a\, \langle\varphi,\, u\rangle + b\, \langle\psi,\, u\rangle\big)\, \varphi + \big(c\, \langle\varphi,\, u\rangle + d\, \langle\psi,\, u\rangle\big)\, \psi.$$

Give an example of real numbers a, b, c, d such that $\sigma(A)$ is purely imaginary.

(h) Define $f \in \mathcal{S}^*(\mathbb{R})$ via $f(x) = x^3$. What is $\hat{f}$? (Do not worry about getting constant multipliers correct.)

Problem 2: Set $I = [0,\, 1]$ and let (f_n) be a sequence of real valued functions in $L^1(I)$ such that

$$\int_0^1 f_n(x)\, dx = 2^{-n},$$

and such that the sum $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is absolutely summable for any $x \in I$.

(a) Prove that if $f_n(x) \geq 0$ for all x and n, then $\int_0^1 f(x)\, dx = 1$. (6p)

(b) Construct functions f_n (not necessarily non-negative) such that $\int_0^1 f(x)\, dx = 0$. (6p)

Problem 3: Set $H = l^2(\mathbb{N})$. Define $A \in \mathcal{B}(H)$ via

$$A(x_1,\, x_2,\, x_3,\, x_4,\, \dots) = (0,\, \frac{x_1}{2},\, \frac{x_2}{3},\, \frac{x_3}{4},\, \dots).$$

Determine $\sigma(A)$, $\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{c}}(A)$, and $\sigma_{\mathrm{r}}(A)$. (10p) *Hint: Calculate powers of A.*

Problem 4: Let α be a real number and define the function f via

$$f(x) = \begin{cases} 0 & x = 0 \\ |x|^\alpha & x \neq 0. \end{cases}$$

For which real numbers α is it the case that $f \in \mathcal{S}^*(\mathbb{R}^d)$? Prove your assertions. (10p)

Problem 5: Let $(\varphi_j)_{j=1}^\infty$ be an ON-basis for a Hilbert space H. Define for $n = 1, 2, 3, \dots$ operators $A_n \in \mathcal{B}(H)$ via

$$A_n\, u = \sum_{j=1}^\infty \left(\frac{1}{j}\right)^{1/n} \langle \varphi_j,\, u\rangle\, \varphi_j.$$

(a) Are the operators A_n self-adjoint? Compact? Motivate your answers briefly. (3p)

(b) Determine the spectrum of A_n and identify $\sigma_{\mathrm{p}}(A_n)$, $\sigma_{\mathrm{c}}(A_n)$, and $\sigma_{\mathrm{r}}(A_n)$.
No motivation required. (3p)

(c) Does the sequence $(A_n)_{n=1}^\infty$ converge? If so, in what sense and to what limit? (6p)