Homework 4

8.15) Prove that for all $A, B \in B(H)$ and $\lambda \in C$ we have:

a) $A^{**} = A$

For any $x, y \in H$ we have $(Ax, y) = (x, A^* y) = ((A^*)^* x, y) = (A^{**} x, y)$
The first and second equalities use the definition of the adjoint and the final equality is notation. Since this holds for all $x, y \in H$ we conclude that $A^{**} = A$.

b) $(AB)^* = B^* A^*$

For any $x, y \in H$ we have $(x, (AB)^* y) = (ABx, y) = (Bx, A^* y) = (x, B^* A^* y)$
All three equalities use the definition of the adjoint. Since this holds for all $x, y \in H$ we conclude that $(AB)^* = B^* A^*$.

c) $(\lambda A)^* = \overline{\lambda} A^*$

For any $x, y \in H$ we have $(\lambda Ax, y) = \lambda (Ax, y) = \lambda (x, A^* y) = (x, \overline{\lambda} A^* y)$
The first and final equality use the linearity and antilieartity of the inner product (respectively) and the middle equality uses the definition of the adjoint. Since this holds for all $x, y \in H$ we conclude that $(\lambda A)^* = \overline{\lambda} A^*$.

d) $(A + B)^* = A^* + B^*$

For any $x, y \in H$ we have

$$(x, (A+B)^* y) = ((A+B)x, y) \overset{L}{=} (Ax, y) + (Bx, y) = (x, A^* y) + (x, B^* y) \overset{L}{=} (x, (A^* + B^*) y)$$

The equalities denoted by "L" use the linearity of the inner product and the others use the definition of the adjoint. Since this holds for all $x, y \in H$ we conclude that $(A + B)^* = A^* + B^*$.

e) $\|A^*\| = \|A\|$

$$\|A\|^2 \overset{N}{=} \sup_{\|x\|=1} (Ax, Ax) \overset{A}{=} \sup_{\|x\|=1} (x, A^* Ax) \overset{N}{=} \|A^* A\| \le \|A^*\| \cdot \|A\| \Rightarrow \|A\| \le \|A^*\|$$

$$\|A^*\|^2 \overset{N}{=} \sup_{\|x\|=1} (A^* x, A^* x) \overset{A}{=} \sup_{\|x\|=1} (x, A^{**} A^* x) \overset{(a)}{=} \sup_{\|x\|=1} (x, AA^* x) \overset{N}{=} \|AA^*\| \le \|A\| \cdot \|A^*\| \Rightarrow \|A^*\| \le \|A\|$$

The equalities denoted by "N" use the definition of the norm, the equalities denoted by "A" use the definition of the adjoint, and the equality denoted by "(a)" uses the property established in part (a) above.

Then: $\|A\| \le \|A^*\| \quad and \quad \|A^*\| \le \|A\| \quad \Rightarrow \quad \|A\| = \|A^*\|$

Problem 1) Consider the Hilbert space $H = l^2(\mathrm{N})$ and let e_n denote the canonical basis vectors. Which of the following converge weakly? Which have convergent subsequences?

Before we start note the following theorem: For an orthonormal basis (e_α):

$$x_n \xrightarrow{weakly} x \Leftrightarrow \begin{cases} \sup\limits_n \|x_n\| < \infty \\ (e_\alpha, x_n) \to (e_\alpha, x) \quad \forall \alpha \end{cases}$$

a) $x_n = ne_n$

Check the first condition: $\sup\limits_n \|x_n\| = \sup\limits_n \|ne_n\| = \sup\limits_n (n) = \infty$

Since this sequence is unbounded it cannot converge weakly. Also, it is clear that any subsequence will also diverge to infinity so it cannot have a convergent subsequence.

b) $x_n = n^{-1/2} \sum\limits_{j=1}^{n} e_j$

First condition: $\|x_n\| = \sum\limits_{j=1}^{n} \left|(x_n, e_j)\right|^2 = \sum\limits_{j=1}^{n} \left|\frac{1}{n^{1/2}}\right|^2 = \sum\limits_{j=1}^{n} \frac{1}{n} = \frac{1}{n} \sum\limits_{j=1}^{n} 1 = \frac{n}{n} = 1$ (it is bounded)

Second condition: $(e_\alpha, x_n) \overset{*}{=} \frac{1}{\sqrt{n}} \to 0$ (it converges "by component")

* Note that the starred equality holds for all $n \geq \alpha$

So this sequence converges weakly (and hence it has convergent subsequences too).

c) $x_n = e_n + e_m$ where $m = 1 + \mathrm{mod}(n, 2)$

Let's write out the first few terms of this sequence:

$x_1 = (2,0,0,0,0,\cdots), \quad x_2 = (0,1,0,0,0,\cdots), \quad x_3 = (1,0,1,0,0,\cdots), \quad x_4 = (0,0,0,1,0,\cdots), \quad \cdots$

First condition: $\|x_n\| = \begin{cases} 1 & n \text{ even} \\ \sqrt{2} & n \text{ odd} \\ 2 & n = 1 \end{cases}$ (it is bounded)

Second condition:

Then: $(e_\alpha, x_n) \to 0 \quad \forall \alpha \neq 1$

and $(e_1, x_n) = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$ (We are only interested in the limit so assume n > 1)

This clearly doesn't converge to anything, but does have convergent subsequences (take n even OR n odd).

So this sequence does not converge weakly, but a subsequence consisting only of the even or only of the odd terms will converge.

Problem 2) Consider the Hilbert space $H = L^2([-\pi,\pi])$ and the sequence $\varphi_n(x) = x^2\sin(nx)$. Does $(\varphi_n)_{n=1}^{\infty}$ converge strongly in H? Does $(\varphi_n)_{n=1}^{\infty}$ converge weakly in H? If the answer to either question is yes then specify the limit.

Weakly) Begin by noting the theorem: $x_n \xrightarrow{weakly} x \Leftrightarrow \begin{cases} \sup\limits_n \|x_n\| < \infty \\ (e_\alpha, x_n) \to (e_\alpha, x) \quad \forall\alpha \end{cases}$

First condition: $\|\varphi_n\|^2 \le \int_{-\pi}^{\pi} \left|x^2\sin(nx)\right|^2 dx = \int_{-\pi}^{\pi} x^4 |\sin(nx)|^2 dx \le \int_{-\pi}^{\pi} x^4 dx \le \frac{2\pi^5}{5}$ (it is bounded)

Second condition: Define $e_m = \frac{e^{imx}}{\sqrt{2\pi}}$ (this is an orthonormal basis). Then for $|n| > |m|$ we have:

$$2i\sqrt{2\pi}(e_m,\varphi_n) = \int_{-\pi}^{\pi} e^{-imx}x^2\left(e^{inx} - e^{-inx}\right)dx = \int_{-\pi}^{\pi} x^2\left(e^{i(n-m)x} - e^{i(-n-m)x}\right)dx = \int_{-\pi}^{\pi} x^2 e^{i(n-m)x}dx - \int_{-\pi}^{\pi} x^2 e^{-i(n+m)x}dx =$$

$$= \int_{-\pi}^{\pi} \frac{2xe^{i(n-m)x}}{i(n-m)}dx - \int_{-\pi}^{\pi} \frac{2xe^{-i(n+m)x}}{i(n+m)}dx + BT \le \frac{1}{(n-m)}\int_{-\pi}^{\pi} |-2ix|\cdot\underbrace{\left|e^{i(n-m)x}\right|}_{\le 1}dx + \frac{1}{(n+m)}\int_{-\pi}^{\pi} |-2ix|\cdot\underbrace{\left|e^{-i(n+m)x}\right|}_{\le 1}dx =$$

$$\le \frac{4\pi}{(n-m)} + \frac{4\pi}{(n+m)} \xrightarrow{n\to\infty} 0$$

The equality across the first line break uses integration by parts (and the arising boundary terms (BT) are zero dues to the periodicity of the space).

Assuming $(\varphi_n)_{n=1}^{\infty}$ is bounded (we showed it is), then $\varphi_n \xrightarrow{weakly} \varphi \Leftrightarrow (e_\alpha,\varphi_n) \to (e_\alpha,\varphi) \quad \forall\alpha$

Thus $(\varphi_n)_{n=1}^{\infty}$ converges weakly to 0.

Strongly) Since $(\varphi_n)_{n=1}^{\infty}$ converges weakly to zero we know that if $(\varphi_n)_{n=1}^{\infty}$ converges strongly then the limit must be zero (i.e. $\|\varphi_n\| \to 0$). However:

$$\|\varphi_n\|^2 = \int_{-\pi}^{\pi} x^4\sin^2(nx)dx = \frac{\pi^5}{5} - \underbrace{\frac{\pi(2n^2\pi^2 - 3)}{2n^4}}_{\xrightarrow{n\to\infty} 0} \xrightarrow{n\to\infty} \frac{\pi^5}{5} \ne 0$$

Since this does not converge to zero we conclude that $(\varphi_n)_{n=1}^{\infty}$ does not converge strongly.

Problem 3) Let A denote a self-adjoint operator on a Hilbert space H. Let $u \in H$ and set $u_n = e^{inA}u$. Prove that $(u_n)_{n=1}^{\infty}$ has a weakly convergent subsequence.

$\|u_n\| = \left\| \underbrace{e^{inA}}_{unitary} u \right\| = \|u\|$ (Note that e^{inA} is unitary as long as A is self-adjoint)

This norm gives us: $u_n \in B_{\|u\|}(0)$. By Theorem 8.45 (Banach-Alaoglu) we know that a closed ball in a Hilbert space is compact in the weak topology, and hence we can conclude that it has a weakly convergent subsequence.

Problem 4) Let H_1 and H_2 be Hilbert spaces. Let $U : H_1 \to H_2$ be a unitary operator and let $A_1 \in B(H_1)$ be a self-adjoint operator. Define the operator $A_2 \in B(H_2)$ by $A_2 = UA_1U^{-1}$. Prove that A_2 is self-adjoint.

For any $x, y \in H_2$ we have:

$$(x, A_2 y)_{H_2} = (x, UA_1U^{-1}y)_{H_2} \overset{bijective}{=} (Ux', UA_1 y')_{H_2} \overset{unitary}{=} (x', A_1 y')_{H_1} \overset{SA}{=} (A_1 x', y')_{H_1} =$$

$$= (A_1 U^{-1}x, U^{-1}y)_{H_1} \overset{unitary}{=} (UA_1U^{-1}x, UU^{-1}y)_{H_2} = (A_2 x, y)_{H_2}$$

The first equality substitutes in the definition of A_2.

The equality denoted by "bijective" uses the fact that (since U is bijective) $\exists x', y' \in H_1$ s.t.

$x = Ux', y = Uy'$.

The equalities denoted by "unitary" use the definition of a unitary operator.

The equality denoted by "SA" uses the fact that A_1 is self-adjoint.

The equality across the line break substitutes $x' = U^{-1}x, y' = U^{-1}y$ back in.

The final equality substitutes in the definition of A_2 and uses the fact that $UU^{-1} = I$.