

Homework set 9 — APPM5450 — Spring 2008

From the textbook: 11.5, 11.9, 11.15.

Problem 1: We say that a sequence $(\varphi_n)_{n=1}^{\infty}$ is an *approximate identity* if

- (1) $\varphi_n \in C(\mathbb{R}^d)$, $\forall n$,
- (2) $\varphi_n(x) \geq 0$, $\forall n, x$,
- (3) $\int_{\mathbb{R}^d} \varphi_n(x) dx = 1$, $\forall n$,
- (4) $\forall \varepsilon > 0$, $\int_{|x| \geq \varepsilon} \varphi_n(x) dx \rightarrow 0$ as $n \rightarrow \infty$.

- (a) Do the conditions imply that $\varphi_n \in \mathcal{S}^*$?
- (b) Assuming that $\varphi_n \in \mathcal{S}^*$, prove that $\varphi_n \rightarrow \delta$ in \mathcal{S}^* .

Problem 2: Compute the Fourier transforms of $f(x) = \chi_{[-R,R]}(x)$ and $f(x) = e^{-a|x|}$ by simply evaluating the formula

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} f(x) dx.$$

The answers are given in examples 11.32 and 11.33 in the text book.

Problem 3 (optional): Let k be a positive integer. Prove that there exist numbers c_k and C_k such that $0 < c_k \leq C_k < \infty$, and

$$(1) \quad c_k (1 + |x|^k) \leq (1 + |x|^2)^{k/2} \leq C_k (1 + |x|^k), \quad \forall x \in \mathbb{R}^d.$$

Check to see if you can readily adapt your proof to also prove the existence of numbers b_k and B_k such that $0 < b_k \leq B_k < \infty$ such that

$$(2) \quad b_k (1 + |x|)^k \leq (1 + |x|^2)^{k/2} \leq B_k (1 + |x|)^k, \quad \forall x \in \mathbb{R}^d.$$

Note 1: The existence of inequalities such as (1) and (2) are routinely used (generally without even commenting on it) to replace the growth factor $(1 + |x|^2)^{k/2}$ in the norms $\|\cdot\|_{\alpha,k}$ by either $(1 + |x|^k)$ or $(1 + |x|)^k$, whenever convenient.

Note 2: If you have time, you may find it interesting to see what happens to the numbers b_k, B_k, c_k, C_k as k grows large. (This is easily done using Matlab.)