Applied Analysis (APPM 5450): Midterm 2 — Solutions
11.35am – 12.50pm, Mar 19, 2008. Closed books.

Problem 1: Let H_1 and H_2 be Hilbert spaces, and let $A \in \mathcal{B}(H_1)$. Suppose further that $U \in \mathcal{B}(H_1, H_2)$ is a unitary map.

(a) Define the following sets: $\rho(A)$, $\sigma(A)$, $\sigma_{\rm p}(A)$, $\sigma_{\rm c}(A)$, $\sigma_{\rm r}(A)$. (4p)

(b) Prove that if $\lambda \in \sigma_{\rm r}(A)$, then $\bar{\lambda} \in \sigma_{\rm p}(A^*)$. (3p)

(c) Define the operator $\hat{A} \in \mathcal{B}(H_2)$ by $\hat{A} = U\,A\,U^{-1}$. Prove that $\sigma_{\rm p}(A) = \sigma_{\rm p}(\hat{A})$. (2p)

(d) Define the operator $\hat{A} \in \mathcal{B}(H_2)$ by $\hat{A} = U\,A\,U^{-1}$. Prove that $\sigma_{\rm c}(A) = \sigma_{\rm c}(\hat{A})$. (2p)

Solution:

(a) See the text book — Definitions 9.3 and 9.4.

(b) See the text book — Proposition 9.12.

(c) Note that

$$\hat{A} - \lambda\,I = U\,A\,U^{-1} - \lambda\,U\,U^{-1} = U\,(A - \lambda\,I)\,U^{-1}. \tag{1}$$

Since U and U^{-1} are both one-to-one, it follows that:

$$\lambda \notin \sigma_{\rm p}(A) \quad \Leftrightarrow \quad \ker(A - \lambda\,I) = \{0\} \quad \Leftrightarrow \quad \ker(\hat{A} - \lambda\,I) = \{0\} \quad \Leftrightarrow \quad \lambda \notin \sigma_{\rm p}(\hat{A})$$

(d) Suppose that $\lambda \in \sigma_{\rm c}(A)$. We will prove that then $\lambda \in \sigma_{\rm c}(\hat{A})$.

Since $\lambda \in \sigma_{\rm c}(A)$, we know that $A - \lambda\,I$ is one-to-one. That $\hat{A} - \lambda I$ is one-to-one then follows from (1) and the fact that U and U^{-1} are one-to-one.

To prove that $\mathrm{ran}(\hat{A} - \lambda I)$ is dense in H_2, pick any $\hat{x} \in H_2$ and any $\varepsilon > 0$. Set $x = U^{-1}\,\hat{x}$. Since $\mathrm{ran}(A - \lambda I)$ is dense in H_1, there exists a $z \in H_1$ such that $||(A - \lambda\,I)\,z - x|| < \varepsilon$. Set $\hat{z} = U\,z$. Then

$$||(\hat{A} - \lambda\,I)\,\hat{z} - \hat{x}|| = ||U\,(A - \lambda\,I)\,U^{-1}\,\hat{z} - U\,x|| = ||U\,\big((A - \lambda\,I)\,z - x\big)|| = ||(A - \lambda\,I)\,z - x|| < \varepsilon.$$

We have proved that $\sigma_{\rm c}(A) \subseteq \sigma_{\rm c}(\hat{A})$. The proof that $\sigma_{\rm c}(\hat{A}) \subseteq \sigma_{\rm c}(A)$ is analogous.

Problem 2: Let $\delta \in \mathcal{S}^*(\mathbb{R})$ denote the Dirac δ-function. Define $T \in \mathcal{S}^*(\mathbb{R})$ via $T(x) = \sin(n\,x)\,\delta'(x)$ where n is an integer, and define $\varphi \in \mathcal{S}(\mathbb{R})$ via $\varphi(x) = (A + B\,x)\,e^{-x^2}$ where A and B are real numbers. Evaluate $\langle \delta',\, \varphi\rangle$ and $\langle T,\, \varphi\rangle$. (5p)

Solution:

$$\langle \delta',\, \varphi\rangle = -\langle \delta,\, \varphi'\rangle = -\varphi'(0) = -B.$$

$$\begin{aligned}\langle T,\, \varphi\rangle = \langle \sin(nx)\,\delta',\, \varphi\rangle = \langle \delta',\, \sin(nx)\,\varphi\rangle = -\langle \delta,\, \frac{d}{dx}\big(\sin(nx)\,\varphi\big)\rangle = \\ -\langle \delta,\, n\,\cos(nx)\,\varphi + \sin(nx)\,\varphi'\rangle = -n\,\varphi(0) = -n\,A.\end{aligned}$$

Problem 3: Set $H = L^2(I)$ where $I = [-1,\,1]$ and let ψ be the function

$$\psi(x) = \begin{cases} -1 & x = -1 \\ 1 + x & x \in (-1,\,0) \\ 1 & x \in [0,\,1]. \end{cases}$$

Define $A \in \mathcal{B}(H)$ by $[A\,u](x) = \psi(x)\,u(x)$. Draw a graph of ψ. Determine $\sigma(A)$, $\sigma_{\rm p}(A)$, $\sigma_{\rm c}(A)$, and $\sigma_{\rm r}(A)$. No motivation required. (8p)

Solution: The answer is:

$\sigma(A) = [0,\,1]$

$\sigma_{\rm p}(A) = \{1\}$

$\sigma_{\rm c}(A) = [1,\,0)$

$\sigma_{\rm r}(A) = \emptyset$

A (non-required) motivation:

If $\lambda \notin [0,\,1]$, then the operator T defined by $[T\,u](x) = \frac{1}{\psi(x)-\lambda}\,u(x)$ is a bounded linear operator that is the inverse of $A - \lambda I$. (Note that it does not matter that $1/(\psi(x)+1)$ blows up at a single point when $\lambda = -1$ since an L^2 function does not change when its value is changed at a single point.) It follows that $\sigma(A) \subseteq [0,\,1]$.

Next we determine $\sigma_{\rm p}(A)$. If u satisfies the equation $(A - \lambda\,I)\,u = 0$, then

$$(\psi(x) - \lambda)\,u(x) = 0. \tag{2}$$

If $\lambda \neq 1$, then (2) implies that $u = 0$ (except for possibly at a single point, but again, this does not change an L^2 function) so $\lambda \notin \sigma_{\rm p}(A)$. If $\lambda = 1$, then any function that is supported in the interval $[0,\,1]$ satisfies (2). It follows that $\sigma_{\rm p}(A) = \{1\}$.

We will finally prove that if $\lambda \in [0,\,1)$, then $\lambda \in \sigma_{\rm c}(A)$. We have already proven that then $\lambda \notin \sigma_{\rm p}(A)$ so $A - \lambda\,I$ is one-to-one. To see that $A - \lambda I$ is not onto, simply note that the constant function 1 belongs to $L^2(I)$, but the equation $(\psi(x) - \lambda)\,u(x) = 1$ does not have a solution $u \in L^2(I)$. To finally prove that $(A - \lambda\,I)$ is dense, note that for any $\varepsilon > 0$, the set $H_\varepsilon = \{u \in H :\ u(x) = 0 \text{ when } |x - \lambda| \leq \varepsilon\}$ belongs to $\text{ran}(A - \lambda I)$, and that

$$\overline{\bigcup_{n=1}^{\infty} H_{1/n}} = L^2(I).$$

Problem 4: Let A be a bounded self-adjoint operator on a Hilbert space A. Consider the following statements:

(a) If $\lambda \in \sigma(A)$, then the imaginary part of λ is zero.

(b) The residual spectrum of A is empty.

(c) If M is an invariant subspace of A, then so is $M^{\perp}$.

(d) The continuous spectrum of A is either empty or consists of the single point 0.

(e) $||A|| = \sup_{||x||=1} |\langle A\,x,\, x\rangle|$.

(f) If λ and μ are two different eigenvalues of A, then $\ker(A - \lambda\,I) \subseteq \big(\ker(A - \mu\,I)\big)^{\perp}$.

For each of the six statements, mark whether it is true or false. (2p) for each correct answer.

Extra credit: Pick ***at most two*** *of the statements (4a) – (4f) and either prove them, or give a counterexample. (2p) for each correct proof/counterexample.*

Solution:

(a) TRUE. See Lemma 9.13 in the book for the proof.

(b) TRUE. If $\lambda \in \sigma_{\rm r}(A)$, then (see Problem 1b!) $\bar{\lambda} \in \sigma_{\rm p}(A^*) = \sigma_{\rm p}(A)$. Since λ must be real, it follows that λ must belong to both $\sigma_{\rm r}(A)$ and $\sigma_{\rm p}(A)$ which is impossible.

(c) TRUE. Suppose that M is an invariant subspace and that $x \in M^{\perp}$. We need to prove that $A\,x \in M^{\perp}$ which is the same as saying that $\langle A\,x,\, y\rangle = 0$ for all $y \in M$. But this must be the case since
(i) $\langle A\,x,\, y\rangle = \langle x,\, A\,y\rangle$
(ii) $A\,y \in M$
(iii) $x \in M^{\perp}$.

(d) FALSE. The operator A in Problem 3 is one counterexample. (Note that if A is self-adjoint *and compact*, then (d) would be true.)

(e) TRUE. See Lemma 8.26 in the book.

(f) TRUE. Suppose that λ and μ are two different eigenvalues, that $u \in \ker(A - \lambda\,I)$, and that $v \in \ker(A - \mu\,I)$. Then $A\,u = \lambda\,u$ and $A\,v = \lambda\,v$. Noting that both λ and μ must be real, we find that

$$\lambda\langle u,\, v\rangle = \langle \lambda\,u,\, v\rangle = \langle A\,u,\, v\rangle = \langle u,\, A\,v\rangle = \langle u,\, \mu\,v\rangle = \mu\langle u,\, v\rangle.$$

It follows that $(\lambda - \mu)\,\langle u,\, v\rangle = 0$, and since $\lambda \neq \mu$, we must have $\langle u,\, v\rangle = 0$.

Problem 5: Consider the map $T : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ defined via $\langle T, \varphi\rangle = \lim_{\varepsilon\searrow 0} \int_{|x|\geq\varepsilon} \frac{1}{x}\,\varphi(x)\,dx$.

(a) Prove that T is continuous. (4p)

(b) Prove that T' is given by $\langle T', \varphi\rangle = \lim_{\varepsilon\searrow 0}\left(-\int_{|x|\geq\varepsilon} \frac{1}{x^2}\,\varphi(x)\,dx + \frac{2\,\varphi(0)}{\varepsilon}\right)$. (4p)

Solution:

(a) First we reformulate the definition of T:

$$\text{(3)} \qquad \langle T, \varphi\rangle = \lim_{\varepsilon\searrow 0} \int_{|x|\geq\varepsilon} \frac{1}{x}\,\varphi(x)\,dx = \lim_{\varepsilon\searrow 0} \int_{\varepsilon}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx = \int_0^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx.$$

Next note that

$$\text{(4)} \qquad \left|\frac{\varphi(x)-\varphi(-x)}{x}\right| = \left|\frac{1}{x}\int_{-x}^{x} \varphi'(y)\,dy\right| \leq \frac{1}{|x|}\int_{-x}^{x} |\varphi'(y)|\,dy \leq \frac{1}{|x|}\,2\,|x|\,||\varphi'||_{\mathrm{u}} = 2\,||\varphi||_{1,0},$$

and that, when $x \geq 1$,

$$\text{(5)} \qquad \left|\frac{\varphi(x)-\varphi(-x)}{x}\right| \leq \frac{1}{|x|}\,\big(|\varphi(x)|+|\varphi(-x)|\big) = \frac{1}{x^2}\,\big(|x\,\varphi(x)|+|x\,\varphi(-x)|\big) \leq \frac{1}{x^2}\,2\,||\varphi||_{0,1}.$$

Combining (3), (4), and (5), we obtain

$$\begin{aligned}|\langle T, \varphi\rangle| &\leq \int_0^1 \left|\frac{\varphi(x)-\varphi(-x)}{x}\right| dx + \int_1^{\infty} \left|\frac{\varphi(x)-\varphi(-x)}{x}\right| dx \\ &\leq \int_0^1 2\,||\varphi||_{1,0}\,dx + \int_1^{\infty} \frac{1}{x^2}\,2\,||\varphi||_{0,1}\,dx = 2\,(||\varphi||_{1,0} + ||\varphi||_{0,1}).\end{aligned}$$

(b) Using the definition of a distributional derivative and partial integration we obtain:

$$\begin{aligned}\langle T', \varphi\rangle = -\langle T, \varphi'\rangle &= -\lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{-\varepsilon} \frac{1}{x}\,\varphi'(x)\,dx + \int_{\varepsilon}^{\infty} \frac{1}{x}\,\varphi'(x)\,dx\right) \\ &= -\lim_{\varepsilon\searrow 0}\left(\left[\frac{1}{x}\,\varphi(x)\right]_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon} \frac{1}{x^2}\,\varphi(x)\,dx + \left[\frac{1}{x}\,\varphi(x)\right]_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty} \frac{1}{x^2}\,\varphi(x)\,dx\right) \\ &= -\lim_{\varepsilon\searrow 0}\left(\frac{\varphi(-\varepsilon)}{-\varepsilon} + \int_{-\infty}^{-\varepsilon} \frac{1}{x^2}\,\varphi(x)\,dx - \frac{\varphi(\varepsilon)}{\varepsilon} + \int_{\varepsilon}^{\infty} \frac{1}{x^2}\,\varphi(x)\,dx\right) \\ &= -\lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{-\varepsilon} \frac{1}{x^2}\,\varphi(x)\,dx + \int_{\varepsilon}^{\infty} \frac{1}{x^2}\,\varphi(x)\,dx - \frac{2\,\varphi(0)}{\varepsilon}\right)\end{aligned}$$