Applied Analysis (APPM 5450): Midterm 3
11.35am – 12.50pm, April 23, 2008. Closed books.

Note: *You may want to save problems marked with a star for last.*

Problem 1: Mark the following as TRUE/FALSE. Motivate your answers briefly.

(a) [2p] If $f_n \rightharpoonup f$ in $L^2(\mathbb{R}^d)$, then $\hat{f}_n \rightharpoonup \hat{f}$ in $L^2(\mathbb{R}^d)$. (Note the *weak* convergence arrows.)

(b) [2p] Set $B = \{f \in L^2(\mathbb{R}^d) : \ ||f||_2 \leq 1\}$. Then $\mathcal{F}$ is a bijection from B to B.

(c) [2p] Let f be a function on $\mathbb{R}$ such that $\displaystyle\int_{-\infty}^{\infty} (1+|x|)\,|f(x)|\,dx < \infty$. Then $\hat{f} \in C^1(\mathbb{R})$.

(d) [2p] If $f_n \to f$ in $L^1(\mathbb{R}^d)$, then $\hat{f}_n \to \hat{f}$ uniformly.

(e) [2p] If $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$ and α is a multi-index, then $\partial^\alpha\,\hat{\varphi}_n \to \partial^\alpha\,\hat{\varphi}$ in $\mathcal{S}(\mathbb{R}^d)$.

Problem 2: [7p] Let d be a positive integer. Prove that if s is a real number that is "large enough", then $H^s(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$. Make sure to specify what "large enough" is.

Problem 3: Calculate the Fourier transform of the following functions on $\mathbb{R}$:

(a) [3p] The Dirac δ-function.

(b) [3p] $f(x) = x^k$.

(c) [3p] $g(x) = \sin(x)$.

Problem 4:

(a) [2p] State the definition of a σ-algebra.

(b) [2p] Is every topology is a σ-algebra? Motivate your answer.

(c*) [2p] Is every σ-algebra a topology? Motivate your answer.

(d) [2p] State the definition of a *measure*.

(e) [4p] Let $(X,\,\mathcal{A},\,\mu)$ be a measure space, and let $\{\Omega_\beta\}_{\beta\in B}$ be a countable collection of sets in $\mathcal{A}$. Prove directly from the definition of a measure that

$$(*)\qquad \mu\left(\bigcup_{\beta\in B}\Omega_\beta\right) = \sup\left\{\mu\left(\bigcup_{\beta\in C}\Omega_\beta\right)\ :\ C \text{ is a finite subset of } B\right\}.$$

Hint: Since B is countable, you may assume that $B = \{1,\,2,\,3,\,\dots\}$. Then the statement you are asked to prove is equivalent to the statement $\mu\left(\bigcup_{n=1}^{\infty}\Omega_n\right) = \sup\left\{\mu\left(\bigcup_{n=1}^{N}\Omega_n\right)\ :\ N = 1,\,2,\,3,\,\dots\right\}$.

(f*) [2p] Demonstrate that the formula (*) is not necessarily true if B is uncountable.

Problem 5: [6p] We define for $n = 1,\,2,\,3,\,\dots$ functions f_n on $\mathbb{R}$ by $f_n(x) = n^{3/2}\,x\,e^{-n\,x^2}$. Either prove that $(f_n)_{n=1}^{\infty}$ does not converges in $\mathcal{S}^*(\mathbb{R})$, or give the limit point and prove convergence.